

## Eigenvalue Problems

1. Determine the eigenvalues and eigenfunctions of the problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0.$$

2. Determine the eigenvalues and eigenfunctions of the problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0.$$

3. Given the operator  $L = -\frac{d^2}{dx^2}$ , we want to determine conditions on  $f$  and  $g$  that make this operator self-adjoint under the inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x) dx,$$

i.e. that

$$\langle L(f), g \rangle = \langle f, L(g) \rangle.$$

- (a) First write down the integral that represents

$$\langle L(f), g \rangle = \left\langle -\frac{d^2}{dx^2}f, g \right\rangle.$$

- (b) Then use integration by parts twice to move the second derivative to  $g$ . Each time you do this there will be a term that needs to be evaluated at the endpoints.

- (c) The extra terms need to be zero for the equality  $\langle L(f), g \rangle = \langle f, L(g) \rangle$  to hold. Write down what must be zero..

(d) Show that the following endpoint conditions satisfy the condition you found in the previous part:

- $f(a) = f(b) = g(a) = g(b) = 0$ . [Dirichlet boundary conditions]

- $f'(a) = f'(b) = g'(a) = g'(b) = 0$ . [Neumann boundary conditions]

- $f(a) = g(a) = 0, hf(b) + f'(a) = 0 = hg(b) + g'(b) = 0$  for  $h > 0$ .