Analytic Solutions

1. For the IVP

$$(1-x)y'' + xy' - y = 0, \quad y(0) = -3, y'(0) = 2$$

(a) Determine the minimum radius of convergence of solutions arond $x_0 = 0$.

(b) If $y = \phi(x)$ is a solution of the IVP, find $\phi''(0), \phi'''(0)$, and $\phi''''(0)$.

(c) Write down the first five terms of the analytic power series solution

$$y = \sum_{n=0}^{\infty} a_n x^n$$

by using the relationship $n!a_n = \phi^{(n)}(x_0)$.

More Power Series

2. Find a general solution to the following differential equation using the power series method.

$$y'' + xy = 0, \quad y(0) = 1, y'(0) = 0.$$

Euler Equations

3. Solve the Euler equation $2x^2y'' + 3xy' - y = 0,\ x > 0$ by looking for solutions of the form $y = x^r$.

   (a) Use the Wronskian to show that the two solutions are linearly independent for $x > 0$.

4. For the Euler equation,

$$x^2y'' + 5xy' + 4y = 0, \quad x > 0,$$

(a) Find one solution $y_1(x)$ by making the substitution $y = x^r$.

(b) Use the method of reduction of order to find the other solution:

i. Assume a second solution of the form $y_2(x) = u(x)y_1(x)$. Plug into the differential equation and simplify to an equation involving u'' and u'.

ii. Solve for u',

iii. Antidifferentiate to determine u.