Eigenvalue Problems

1. Determine the eigenvalues and eigenfunctions of the problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0.$$

2. Determine the eigenvalues and eigenfunctions of the problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0.$$

3. Given the operator $L = -\frac{d^2}{dx^2}$, we want to determine conditions on f and g that make this operator self-adjoint under the inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx,$$

i.e. that

$$\langle L(f), g \rangle = \langle f, L(g) \rangle.$$

(a) First write down the integral that represents

$$\langle L(f), g \rangle = \left\langle -\frac{d^2}{dx^2} f, g \right\rangle.$$

(b) Then use integration by parts twice to move the second derivative to g. Each time you do this there will be a term that needs to be evaluated at the end-points.

(c) The extra terms need to be zero for the equality $\langle L(f), g \rangle = \langle f, L(g) \rangle$ to hold. Write down what must be zero..

(d) Show that the following endpoint conditions satisfy the condition you found in the previous part:

- $f(a) = f(b) = g(a) = g(b) = 0$. [Dirichlet boundary conditions]

- $f'(a) = f'(b) = g'(a) = g'(b) = 0$. [Neumann boundary conditions]

- $f(a) = g(a) = 0, hf(b) + f'(a) = 0 = hg(b) + g'(b) = 0$ for $h > 0$.