Eigenvalue Problems

1. Determine the eigenvalues and eigenfunctions of the problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0.$$

$r^2 + \lambda = 0$

$r = \pm\sqrt{-\lambda}$

$\lambda > 0$

$= \pm\sqrt{\lambda}\, i$

$y(x) = c_1 \cos\sqrt{\lambda}\, x + c_2 \sin\sqrt{\lambda}\, x$

$y(0) = 0 \Rightarrow c_1 = 0$

$y(\pi) = 0 \Rightarrow \sqrt{\lambda} = \frac{n\pi}{\pi}$

$\boxed{\lambda_n = n^2}$ for $n = 1, 2, 3, \ldots$

$\boxed{y_n = \sin nx}$

$\lambda = 0$

$y = c_1 + c_2 x$

$y(0) = c_1 = 0 \Rightarrow y = 0 \Rightarrow$ no eigenfunctions

$y(\pi) = c_2\pi = 0$

$\lambda < 0$

$y = c_1 e^{-\sqrt{-\lambda}\, x} + c_2 e^{+\sqrt{-\lambda}\, x} = c_1 \cosh\sqrt{-\lambda}\, x + c_2 \sinh(-\sqrt{\lambda}\, x)$

$y(0) = c_1 = 0$

$y(\pi) = c_2 \sinh(-\sqrt{\lambda}\, x) = 0 \Rightarrow c_2 = 0 \Rightarrow$ no eigenfunctions

2. Determine the eigenvalues and eigenfunctions of the problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0.$$

As in #1, it is the case that $\lambda > 0$ is the only case that produces non-trivial solutions

$$r^2 + \lambda = 0$$

$$r = \pm\sqrt{\lambda}\, i$$

$$y(x) = c_1 \cos\sqrt{\lambda}x + c_2 \sin\sqrt{\lambda}\, x$$

$$y(0) = c_1 = 0$$

$$y'(L) = c_2\sqrt{\lambda}\cos\sqrt{\lambda}L = 0$$

$$L\sqrt{\lambda} = n\pi - \frac{\pi}{2}$$

$$\boxed{\lambda_n = \left(\frac{n\pi}{L} - \frac{\pi}{2L}\right)^2}$$

$$\boxed{y_n = \sin\left(n\pi - \frac{\pi}{2}\right)\frac{x}{L}}$$

3. Given the operator $L = -\frac{d^2}{dx^2}$, we want to determine conditions on f and g that make this operator self-adjoint under the inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x)\,dx,$$

i.e. that

$$\langle L(f), g \rangle = \langle f, L(g) \rangle.$$

(a) First write down the integral that represents

$$\langle L(f), g \rangle = \left\langle -\frac{d^2}{dx^2} f, g \right\rangle.$$

$$\int_a^b \left(-\frac{d^2}{dx^2} f\right) g \, dx = \int_a^b -f''(x)\, g(x)\, dx$$

(b) Then use integration by parts twice to move the second derivative to g. Each time you do this there will be a term that needs to be evaluated at the end-points.

$$\langle L(f), g \rangle = -f'g\Big|_a^b + \int_a^b f'g'\,dx$$

$$= -f'(b)g(b) + f'(a)g(a) + fg'\Big|_a^b - \int_a^b f g''\,dx$$

$$= \begin{matrix} -f'(b)g(b) + f'(a)g(a) \\ + f(b)g'(b) - f(a)g'(a) \end{matrix} + \int_a^b f(-g'')\,dx$$

(c) The extra terms need to be zero for the equality $\langle L(f), g \rangle = \langle f, L(g) \rangle$ to hold. Write down what must be zero..

$$\begin{matrix} f(b)g'(b) - f'(b)g(b) \\ + f'(a)g(a) - f(a)g'(a) \end{matrix} = 0$$

(d) Show that the following endpoint conditions satisfy the condition you found in the previous part:

- $f(a) = f(b) = g(a) = g(b) = 0$. [Dirichlet boundary conditions]

$$f(b)g'(b) - f'(b)g(b) + f'(a)g(a) - f(a)g'(a) = 0 \cdot g'(b) - f'(b) \cdot 0 + f'(a) \cdot 0 - 0 \cdot g'(a)$$

$$= 0 \checkmark$$

- $f'(a) = f'(b) = g'(a) = g'(b) = 0$. [Neumann boundary conditions]

Inspection

- $f(a) = g(a) = 0, hf(b) + f'(a) = 0 = hg(b) + g'(b) = 0$ for $h > 0$.

$$f(b)g'(b) - f'(b)g(b) + f'(a)\cancel{g(a)}^{0} - \cancel{f(a)}^{0}g'(a) = \left(-\frac{f'(a)}{h}\right)g'(b) + f'(b)\left(+\frac{g'(b)}{h}\right)$$

$$= \frac{f'(b)\,g'(b)}{h} - \frac{f'(b)\,g'(b)}{h}$$

$$= 0 \checkmark$$