

Eigenvalue Problems

1. Determine the eigenvalues and eigenfunctions of the problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(\pi) = 0.$$

$$r^2 + \lambda = 0$$

$$r = \pm \sqrt{-\lambda}$$

$\lambda > 0$

$$= \pm \sqrt{\lambda} i$$

$$y(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x$$

$$y(0) = 0 \Rightarrow c_1 = 0$$

$$y(\pi) = 0 \Rightarrow \sqrt{\lambda} = \frac{n\pi}{1}$$

$$\lambda_n = n^2$$

for $n = 1, 2, 3, \dots$

$$y_n = \sin nx$$

$\lambda = 0$

$$y = c_1 + c_2 x$$

$$y(0) = c_1 = 0$$

$$y(\pi) = c_2 \pi = 0$$

$\Rightarrow y = 0 \Rightarrow$ no eigenfunctions

$\lambda < 0$

$$y = c_1 e^{-\sqrt{\lambda} x} + c_2 e^{+\sqrt{\lambda} x} = c_1 \cosh \sqrt{\lambda} x + c_2 \sinh(-\sqrt{\lambda} x)$$

$$y(0) = c_1 = 0$$

$$y(\pi) = c_2 \sinh(-\sqrt{\lambda} \pi) = 0 \Rightarrow c_2 = 0$$

\Rightarrow no eigenfunctions

2. Determine the eigenvalues and eigenfunctions of the problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0.$$

As in #1, it is the case that $\lambda > 0$ is the only case that produces non-trivial solutions

$$r^2 + \lambda = 0$$

$$r = \pm \sqrt{\lambda} \text{ ;}$$

$$y(x) = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x$$

$$y(0) = c_1 = 0$$

$$y'(L) = c_2 \sqrt{\lambda} \cos \sqrt{\lambda} L = 0$$

$$L \sqrt{\lambda} = n\pi - \frac{\pi}{2}$$

$$\lambda_n = \left(\frac{n\pi}{L} - \frac{\pi}{2L} \right)^2$$

$$y_n = \sin \left(n\pi - \frac{\pi}{2} \right) \frac{x}{L}$$

3. Given the operator $L = -\frac{d^2}{dx^2}$, we want to determine conditions on f and g that make this operator self-adjoint under the inner product

$$\langle f, g \rangle = \int_a^b f(x)g(x) dx,$$

i.e. that

$$\langle L(f), g \rangle = \langle f, L(g) \rangle.$$

- (a) First write down the integral that represents

$$\langle L(f), g \rangle = \left\langle -\frac{d^2}{dx^2}f, g \right\rangle.$$

$$\int_a^b \left(-\frac{d^2}{dx^2}f \right) g dx = \int_a^b -f''(x)g(x) dx$$

- (b) Then use integration by parts twice to move the second derivative to g . Each time you do this there will be a term that needs to be evaluated at the endpoints.

$$\begin{aligned} \langle L(f), g \rangle &= -f'g \Big|_a^b + \int_a^b f'g' dx \\ &= -f'(b)g(b) + f'(a)g(a) + f g' \Big|_a^b - \int_a^b f g'' dx \\ &= -f'(b)g(b) + f'(a)g(a) + f(b)g'(b) - f(a)g'(a) + \int_a^b f(-g'') dx \end{aligned}$$

- (c) The extra terms need to be zero for the equality $\langle L(f), g \rangle = \langle f, L(g) \rangle$ to hold. Write down what must be zero..

$$\begin{aligned} f(b)g'(b) - f'(b)g(b) \\ + f'(a)g(a) - f(a)g'(a) \end{aligned} = 0$$

(d) Show that the following endpoint conditions satisfy the condition you found in the previous part:

- $f(a) = f(b) = g(a) = g(b) = 0$. [Dirichlet boundary conditions]

$$\begin{aligned} f(b)g'(b) - f'(b)g(b) &= 0 \cdot g'(b) - f'(b) \cdot 0 \\ + f'(a)g(a) - f(a)g'(a) &= + f'(a) \cdot 0 - 0 \cdot g'(a) \\ &= 0 \quad \checkmark \end{aligned}$$

- $f'(a) = f'(b) = g'(a) = g'(b) = 0$. [Neumann boundary conditions]

Inspection

- $f(a) = g(a) = 0, hf(b) + f'(a) = 0 = hg(b) + g'(b) = 0$ for $h > 0$.

$$\begin{aligned} f(b)g'(b) - f'(b)g(b) &= \left(-\frac{f'(a)}{h}\right)g'(b) + f'(b)\left(+\frac{g'(b)}{h}\right) \\ + f'(a)g(a) - f(a)g'(a) &= \frac{f'(b)g'(b)}{h} - \frac{f'(b)g'(b)}{h} \\ &= 0 \quad \checkmark \end{aligned}$$