The Heat Equation

1. Given the heat equation

\[ u_t = ku_{xx}, \quad 0 < x < 2, \quad t > 0 \]
\[ u(0, t) = u(2, t) = 0, \quad t > 0 \]
\[ u(x, 0) = 1, \quad 0 < x < 2 \]

(a) We will look for all solutions of the form \( u(x, t) = X(x)T(t) \). Plug this into the heat equation to find an eigenvalue equation in \( x \).

(b) Why do the boundary conditions on \( u \), i.e. \( u(0, t) = u(2, t) = 0 \) imply that \( X(0) = X(2) = 0 \)?

(c) Solve the eigenvalue equation in \( x \).
(d) Solve for $T(t)$.

(e) Why can we write $u(x, t) = \sum_n T_n(t)X_n(x)$, where $X_n(x) = \sin\left(\frac{n\pi x}{2}\right)$?
(f) How do we match the solution up to \( u(x, 0) = 1 \)?
2. Solve the heat equation with Dirichlet boundary conditions

\[ u_t = k u_{xx}, \quad 0 < x < 2, \quad t > 0 \]
\[ u_x(0, t) = u_x(2, t) = 0, \quad t > 0 \]
\[ u(x, 0) = f(x), \quad 0 < x < 2 \]