

The Heat Equation

1. Given the heat equation

$$u_t = ku_{xx}, \quad 0 < x < 2, \quad t > 0$$

$$u(0, t) = u(2, t) = 0, \quad t > 0$$

$$u(x, 0) = 1, \quad 0 < x < 2$$

- (a) We will look for all solutions of the form $u(x, t) = X(x)T(t)$. Plug this into the heat equation to find an eigenvalue equation in x .

- (b) Why do the boundary conditions on u , i.e. $u(0, t) = u(2, t) = 0$ imply that $X(0) = X(2) = 0$?

- (c) Solve the eigenvalue equation in x .

(d) Solve for $T(t)$.

(e) Why can we write $u(x, t) = \sum_n T_n(t)X_n(x)$, where $X_n(x) = \sin\left(\frac{n\pi x}{2}\right)$?

(f) How do we match the solution up to $u(x, 0) = 1$?

2. Solve the heat equation with Dirichlet boundary conditions

$$\begin{aligned}u_t &= k u_{xx}, & 0 < x < 2, & \quad t > 0 \\u_x(0, t) &= u_x(2, t) = 0, & t > 0 \\u(x, 0) &= f(x), & 0 < x < 2\end{aligned}$$