The Heat Equation

1. Given the heat equation

$$u_t = ku_{xx}, \quad 0 < x < 2, \quad t > 0$$
$$u(0,t) = u(2,t) = 0, \quad t > 0$$
$$u(x,0) = 1, \quad 0 < x < 2$$

(a) We will look for all solutions of the form $u(x,t) = X(x)T(t)$. Plug this into the heat equation to find an eigenvalue equation in x.

$$XT' = kX''T$$

$$X'' + \left(-\frac{1}{k}\frac{T'}{T}\right)X = 0$$

$$\lambda = -\frac{1}{k}\frac{T'}{T}$$

(b) Why do the boundary conditions on u, i.e. $u(0,t) = u(2,t) = 0$ imply that $X(0) = X(2) = 0$?

$$u(0,t) = X(0)T(t) = 0$$

if $T(t) = 0$, then $u(x,t) = 0 \Rightarrow X(0) = 0$

don't want this sol'n

(c) Solve the eigenvalue equation in x.

$$\boxed{X_n = \sin\frac{n\pi x}{2}}$$

$$\Rightarrow \lambda = \frac{n^2\pi^2}{4} = -\frac{1}{k}\frac{T'}{T}$$

(d) Solve for $T(t)$.

$$\frac{T'}{T} = -\frac{n^2\pi^2}{4}k$$

$$T_n = e^{-\frac{n^2\pi^2}{4}kt}$$

(e) Why can we write $u(x,t) = \sum_n T_n(t)X_n(x)$, where $X_n(x) = \sin\left(\frac{n\pi x}{2}\right)$?

$u(x,t) = X_n(x)\,T_n(t)$ is a solution to homogeneous BVP for any $n > 0$

By linearity, any linear combo. of solutions is also a solution to the BVP

(f) How do we match the solution up to $u(x, 0) = 1$?

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$$u(x,0) = X(x)\,T(0)^{\to 1}$$

$$= \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{2} = 1$$

$$a_n = \frac{\int_0^2 1\cdot \sin\frac{n\pi x}{2}\,dx}{\int_0^2 \left(\sin\frac{n\pi x}{2}\right)^2 dx}$$

$$\frac{1-\cos n\pi x}{2}$$

$$= \frac{\frac{2}{n\pi}(-2)}{\cancel{\frac{4}{2}}\,1} = -\frac{4}{n\pi}$$

$$\boxed{u(x,t) = \sum_{n=1}^{\infty} -\frac{4}{n\pi}\sin\frac{n\pi x}{2}\, e^{-\frac{n^2\pi^2}{4}kt}}$$

2. Solve the heat equation with Dirichlet boundary conditions

$$u_t = ku_{xx}, \quad 0 < x < 2, \quad t > 0$$
$$u_x(0,t) = u_x(2,t) = 0, \quad t > 0$$
$$u(x,0) = f(x), \quad 0 < x < 2$$

let $u(x,t) = X(x)T(t)$

$XT' - kX''T = 0$

$X'' + \left(-\frac{T'}{kT}\right)X = 0$

$X'(0) = X'(2) = 0$

$\Rightarrow X_n = \cos\frac{n\pi x}{2} \quad \Rightarrow \lambda = \frac{n^2\pi^2}{4} = -\frac{T'}{kT}$

$T_n = e^{-\frac{n^2\pi^2}{4}kt}$

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} a_n \left(\cos\frac{n\pi x}{2}\right) e^{-\frac{n^2\pi^2}{4}kt}}$$

where $a_n = \cancel{\frac{2}{L}}^{1} \int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$

$$\boxed{a_n = \int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx}$$