

## Heat Equation

1. Solve the following heat equation.

$$u_t = u_{xx}, \quad 0 < x < 10, \quad 0 < t$$

$$u(0, t) = 20, \quad u(10, t) = 40$$

$$u(x, 0) = 60, \quad 0 < x < 10$$

$$\text{Let } w(x, t) = X(x)T(t), \quad v_{xx}(x) = 0$$

Dirichlet

$$\Rightarrow X = \sin \frac{n\pi x}{10}$$

$$\lambda = \frac{\pi^2 n^2}{100}$$

$$T = a_n e^{-\frac{\pi^2 n^2}{100} t}$$

$$v(0) = 20$$

$$v(10) = 40$$

$$v(x) = ax + b$$

$$= 2x + 20$$

$$u(x, t) = v(x) + w(x, t)$$

$$= 2x + 20 + \sum a_n e^{-\frac{\pi^2 n^2}{100} t} \sin \frac{n\pi x}{10}$$

$$u(x, 0) = 2x + 20 + \sum a_n \sin \frac{n\pi x}{10} = 60$$

$$\sum a_n \sin \frac{n\pi x}{10} = -2x + 40$$

$$a_n = \frac{2}{10} \int_0^{10} (-2x + 40) \sin \frac{n\pi x}{10} dx$$

$$a_n = \frac{(4x - 80)}{n\pi} (\cos n\pi - 1)$$

$$u(x, t) = 2x + 20 + \sum a_n e^{-\frac{\pi^2 n^2}{100} t} \sin \frac{n\pi x}{10}$$

## Wave Equation

2. Solve the wave equation with Dirichlet boundary conditions

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0$$

$$u(0, t) = u(L, t) = 0, \quad t > 0$$

$$u(x, 0) = L(1 - x), \quad u_t(x, 0) = 0, \quad 0 < x < L$$

(a) We can either look for all solutions of the form  $u(x, t) = X(x)T(t)$ , or recognize the boundary conditions as Dirichlet boundary conditions, and look for solutions of the form

$$u(x, t) = \sum_n c_n(t) \phi_n(x).$$

What are the functions  $\phi_n(x)$ ? What eigenvalue problem do they solve?

$$u(x, t) = X(x) T(t)$$

$$X(x) T''(t) = c^2 X''(x) T(t)$$

$$\frac{X''}{X} = \frac{1}{c^2} \frac{T''}{T} = -\lambda$$

$$X'' + \lambda X = 0$$

$$\phi_n(x) = \sin \frac{n\pi x}{L}$$

$$\lambda = \frac{\pi^2 n^2}{L^2}$$

(b) What differential equation must  $c_n(t)$  solve?

$$c_n'' + c^2 \lambda c_n = 0$$

$$c_n''(t) + \left( c^2 \frac{n^2 \pi^2}{L^2} \right) c_n(t) = 0, \quad c_n'(0) = 0$$

(c) Find  $c_n(t)$ .

$$c_n(t) = a_n \cos \frac{cn\pi}{L} t$$

(d) How do we match up the initial conditions  $u(x, 0) = L(1-x)$  and  $u_t(x, 0) = 0$ ?

$$u(x, 0) = \sum_n a_n \sin \frac{n\pi x}{L} \cdot \cos 0 = L(1-x)$$

$$C_n'(t) = 0$$

$$\Rightarrow C_n(t) = \cos \dots$$

$$a_n = \frac{2}{L} \int_0^L L(1-x) \sin \frac{n\pi x}{L} dx$$

1. Solve the following wave equation.

$$u_{tt} = 4u_{xx}, \quad 0 < x < 10, \quad 0 < t$$

$$u_x(0, t) = 0, \quad u_x(10, t) = 0$$

$$u(x, 0) = 40 - 2x, \quad 0 < x < 10$$

$$u_t(x, 0) = 2x - 40, \quad 0 < x < 10$$

$$u = X T$$

$$\frac{X''}{X} = \frac{1}{4} \frac{T''}{T} = -\lambda$$

$$X = \cos \frac{n\pi x}{10}$$

$$T = a_n \cos \frac{2n\pi t}{10} + b_n \sin \frac{2n\pi t}{10}$$

$$a_n = \frac{2}{10} \int_0^{10} (40 - 2x) \cos \frac{n\pi x}{10} dx$$

$$b_n = \frac{2}{10} \cdot \frac{10}{2n\pi} \int_0^{10} (2x - 40) \cos \frac{n\pi x}{10} dx$$

$$u(x, t) = \sum_n \left( \cos \frac{n\pi x}{10} \right) \left( a_n \cos \frac{n\pi t}{5} + b_n \sin \frac{n\pi t}{5} \right)$$