Heat Equation

1. Solve the following heat equation.

$$u_t = u_{xx}, \quad 0 < x < 10, \quad 0 < t$$
$$u(0,t) = 20, \quad u(10,t) = 40$$
$$u(x,0) = 60, \quad 0 < x < 10$$

Let $w(x,t) = X(x)\,T(t)$, $v_{xx}(x) = 0$

Dirichlet

$v(0) = 20$

$v(10) = 40$

$\Rightarrow X = \sin \frac{\pi n x}{10}$

$v(x) = ax + b$

$= 2x + 20$

$\lambda = \frac{\pi^2 n^2}{100}$

$T = a_n e^{-\frac{\pi^2 n^2}{100} t}$

$u(x,t) = v(x) + w(x,t)$

$= 2x + 20 + \sum a_n e^{-\frac{\pi^2 n^2}{100} t} \sin \frac{n\pi x}{10}$

$u(x,0) = 2x + 20 + \sum a_n \sin \frac{n\pi x}{10} = 60$

$\sum a_n \sin \frac{n\pi x}{10} = -2x + 40$

$a_n = \frac{2}{10} \int_0^{10} (-2x + 40) \sin \frac{n\pi x}{10}\, dx$

$\boxed{a_n = \left(\frac{4x - 80}{n\pi}\right)(\cos n\pi - 1)}$

$\boxed{u(x,t) = 2x + 20 + \sum a_n e^{-\frac{\pi^2 n^2}{100} t} \sin \frac{n\pi x}{10}}$

Wave Equation

2. Solve the wave equation with Dirichlet boundary conditions

$$u_{tt} = c^2 u_{xx}, \quad 0 < x < L, \quad t > 0$$
$$u(0,t) = u(L,t) = 0, \quad t > 0$$
$$u(x,0) = L(1-x), \quad u_t(x,0) = 0, \quad 0 < x < L$$

(a) We can either look for all solutions of the form $u(x,t) = X(x)T(t)$, or recognize the boundary conditions as Dirichlet boundary conditions, and look for solutions of the form

$$u(x,t) = \sum_n c_n(t)\phi_n(x).$$

What are the functions $\phi_n(x)$? What eigenvalue problem do they solve?

$u(x,t) = X(x)\,T(t)$

$X(x)\,T''(t) = c^2 X''(x)\,T(t)$

$\dfrac{X''}{X} = \dfrac{1}{c^2}\dfrac{T''}{T} = -\lambda$

$\boxed{X'' + \lambda X = 0}$

$\boxed{\phi_n(x) = \sin \frac{n\pi x}{L}}$

$\lambda = \dfrac{\pi^2 n^2}{L^2}$

(b) What differential equation must $c_n(t)$ solve?

$C_n'' + c^2 \lambda C_n = 0$

$C_n''(t) + \left(c^2 \frac{n^2\pi^2}{L^2}\right) C_n(t) = 0$, $C_n'(0) = 0$

(c) Find $c_n(t)$.

$C_n(t) = a_n \cos \frac{cn\pi}{L} t$

(d) How do we match up the initial conditions $u(x,0) = L(1-x)$ and $\underbrace{u_t(x,0) = 0}$?

$u(x,0) = \sum_1 a_n \sin\frac{n\pi x}{L} \cdot \cancel{\cos 0}^{1} = L(1-x)$

$c_n'(t) = 0$

$\Rightarrow c_n(t) = \cos\ldots$

$$a_n = \frac{2}{L}\int_0^L L(1-x)\sin\frac{n\pi x}{L}\,dx$$

1. Solve the following wave equation.

$$u_{tt} = 4u_{xx}, \quad 0 < x < 10, \quad 0 < t$$
$$u_x(0,t) = 0, \quad u_x(10,t) = 0$$
$$u(x,0) = 40 - 2x, \quad 0 < x < 10$$
$$u_t(x,0) = 2x - 40, \quad 0 < x < 10$$

$u = XT$

$\frac{X''}{X} = \frac{1}{4}\frac{T''}{T} = -\lambda$

$X = \cos\frac{n\pi x}{10}$

$T = a_n \cos\frac{2n\pi t}{10} + b_n \sin\frac{2n\pi t}{10}$

$$\boxed{\begin{aligned} a_n &= \frac{2}{10}\int_0^{10}(40-2x)\cos\frac{n\pi x}{10}\,dx \\ b_n &= \frac{2}{10}\cdot\frac{10}{2n\pi}\int_0^{10}(2x-40)\cos\frac{n\pi x}{10}\,dx \end{aligned}}$$

$$\boxed{u(x,t) = \sum_1 \left(\cos\frac{n\pi x}{10}\right)\left(a_n\cos\frac{n\pi t}{5} + b_n \sin\frac{n\pi t}{5}\right)}$$