Laplace's Equation

1. Given the equation,

$$u_{xx} + u_{yy} = 0, \quad 0 \le x \le a, \quad 0 \le y \le b$$
$$u(x,0) = f(x), u(x,b) = g(x), \quad u(0,y) = h(y), u(a,y) = k(y)$$

   (a) Break the problem into four cases,

   Case 1:

$$u_{xx} + u_{yy} = 0, \quad 0 \le x \le a, \quad 0 \le y \le b$$
$$u(x,0) = f(x), u(x,b) = 0, \quad u(0,y) = 0, u(a,y) = 0$$

   Case 2:

$$u_{xx} + u_{yy} = 0, \quad 0 \le x \le a, \quad 0 \le y \le b$$
$$u(x,0) = 0, u(x,b) = g(x), \quad u(0,y) = 0, u(a,y) = 0$$

   Case 3:

$$u_{xx} + u_{yy} = 0, \quad 0 \le x \le a, \quad 0 \le y \le b$$
$$u(x,0) = 0, u(x,b) = 0, \quad u(0,y) = h(y), u(a,y) = 0$$

   Case 4:

$$u_{xx} + u_{yy} = 0, \quad 0 \le x \le a, \quad 0 \le y \le b$$
$$u(x,0) = 0, u(x,b) = 0, \quad u(0,y) = 0, u(a,y) = k(y)$$

   (b) Show that the sum of the solution to the four cases is a solution to the overall problem.

(c) Solve case 4.

i. First expand $u(x)$ using the Dirichlet bases $\{\phi_n(x)\}$ such that $\phi_n(0) = \phi_n(a) = 0$.

ii. Then solve the differential equation for y using the initial condition $c_n(y) = 0$.

iii. Write down the overall solution and match the initial condition $u(a, y) = k$

(d) Solve case 2.

(e) Solve case 3. The initial condition $u(0, y)$ will be a bit trickier.

(f) Solve case 1.