Laplace's Equation

1. Given the equation,

$$u_{xx} + u_{yy} = 0, \quad 0 \le x \le a, \quad 0 \le y \le b$$
$$u(x,0) = f(x), u(x,b) = g(x), \quad u(0,y) = h(y), u(a,y) = k(y)$$

(a) Break the problem into four cases,

Case 1:

u_1

$$u_{xx} + u_{yy} = 0, \quad 0 \le x \le a, \quad 0 \le y \le b$$
$$u(x,0) = f(x), u(x,b) = 0, \quad u(0,y) = 0, u(a,y) = 0$$

Case 2:

u_2

$$u_{xx} + u_{yy} = 0, \quad 0 \le x \le a, \quad 0 \le y \le b$$
$$u(x,0) = 0, u(x,b) = g(x), \quad u(0,y) = 0, u(a,y) = 0$$

Case 3:

u_3

$$u_{xx} + u_{yy} = 0, \quad 0 \le x \le a, \quad 0 \le y \le b$$
$$u(x,0) = 0, u(x,b) = 0, \quad u(0,y) = h(y), u(a,y) = 0$$

Case 4:

u_4

$$u_{xx} + u_{yy} = 0, \quad 0 \le x \le a, \quad 0 \le y \le b$$
$$u(x,0) = 0, u(x,b) = 0, \quad u(0,y) = 0, u(a,y) = k(y)$$

(b) Show that the sum of the solution to the four cases is a solution to the overall problem.

$u := u_1 + u_2 + u_3 + u_4$

$$u_{xx} + u_{yy} = u_{1,xx} + u_{1,yy} + u_{2,xx} + u_{2,yy} + \cdots$$
$$= 0 + 0 + 0 + 0$$
$$= 0 \checkmark$$

$$u(x,0) = u_1(x,0) + u_2(x,0) + u_3(x,0) + u_4(x,0)$$
$$= f(x) + 0 = f(x) \checkmark$$

same for other BCs

(c) Solve case ~~1~~ 2

i. First expand $u(x)$ using the Dirichlet bases $\{\phi_n(x)\}$ such that $\phi_n(0) = \phi_n(a) = 0$.

$$u(x) = \sum_1 \phi_n c_n$$

$$u_{xx} + u_{yy} = \phi_n'' c_n + \phi_n c_n'' = 0$$

$$\frac{\phi_n''}{\phi_n} = -\frac{c_n''}{c_n} = -\lambda$$

$$\phi_n(x) = \sin\frac{n\pi x}{a}$$

$$\lambda_n = \frac{n^2\pi^2}{a^2}$$

ii. Then solve the differential equation for y using the initial condition $c_n(0) = 0$.

$\cancel{\lambda = 0 \quad c_0(y) = ay + b}$ $\lambda = 0$ makes $\phi = 0 \Rightarrow$ not interesting

$$\lambda_n = \frac{n^2\pi^2}{a^2} : \quad c_n(y) = \sin\frac{n\pi y}{a}$$

(d) Solve case 4.

$$\phi_n(y) = \sin \frac{n\pi y}{b}$$

$$C_n(x) = a_n \sin \frac{n\pi x}{b}$$

$$u(a,y) = \sum_1 a_n \sin \frac{n\pi y}{b} \sin \frac{n\pi a}{b} = k(y)$$

$$a_n = \frac{2}{b \sin \frac{n\pi a}{b}} \int_0^b k(y) \sin \frac{n\pi y}{b} \, dy$$

iii. Write down the overall solution and match the initial condition $u(\overset{x,b}{\cancel{a,y}}) = \cancel{f}\; g$

$$u = \sum_1 a_n \sin\frac{n\pi x}{a} \sin\frac{n\pi y}{a}$$

$$u(x,b) = \sum_1 a_n \sin\frac{n\pi b}{a} \sin\frac{n\pi x}{a} = g(x)$$

$$a_n = \frac{2}{a\sin\frac{n\pi b}{a}} \int_0^a g(x) \sin\frac{n\pi x}{a}\, dx$$

(e) Solve case 3. The initial condition $u(0, y)$ will be a bit trickier.

$$\phi_n(y) = \sin\frac{n\pi y}{b} \qquad C_n(x) = a_n \sin\frac{n\pi x}{b} + b_n \cos\frac{n\pi x}{b}$$

$$C_n(a) = a_n \sin\frac{n\pi a}{b} + b_n \cos\frac{n\pi a}{b} = 0$$

$$b_n = a_n \tan\frac{n\pi a}{b}$$

Easier to say

$$C_n(x) = a_n \sin\left(\frac{n\pi x}{b} - \frac{n\pi a}{b}\right)$$

$$u(0,y) = \sum -a_n \sin\frac{n\pi a}{b} \sin\frac{n\pi y}{b} = h(y)$$

$$a_n = -\frac{2}{b \sin\frac{n\pi a}{b}} \int_0^b h(y) \sin\frac{n\pi y}{b}\, dy$$

$$u(x,y) = \sum_1 a_n \sin\frac{n\pi y}{b} \sin\left(\frac{n\pi(x-a)}{b}\right)$$

(f) Solve case 1.

$$\phi_n(x) = \sin\frac{n\pi x}{a}$$

$$C_n(y) = a_n \sin\left(\frac{n\pi(y-b)}{a}\right)$$

$$u(x,0) = \sum_1 a_n \sin\frac{n\pi b}{a} \sin\frac{n\pi x}{a} = f(x)$$

$$a_n = -\frac{2}{a\sin\frac{n\pi b}{a}} \int_0^a \sin\frac{n\pi x}{a}\, dx$$

$$u(x,y) = \sum_1 a_n \sin\frac{n\pi x}{a} \sin\frac{n\pi(y-b)}{a}$$