

Differential Equation Solution Methods

1. Find a general solution to the following differential equations.

$$(a) \underbrace{4x^3 + 2xy^2}_M + \underbrace{(2x^2y + 4y^3)}_N \frac{dy}{dx} = 0$$

$$N_x = 4xy \quad \Rightarrow \text{Exact}$$

$$M_y = 4xy$$

$$\phi_y = 2x^2y + 4y^3$$

$$\phi_x = 4x^3 + 2xy^2 \quad \Rightarrow \quad \phi = x^2y^2 + x^4 + y^4 = c$$

$$(b) y^3 \frac{dy}{dx} = (y^4 + 1) \cos x$$

$$\frac{1}{4} (y^4 + 1)' = (y^4 + 1) \cos x$$

$$\int \frac{d(y^4 + 1)}{y^4 + 1} = \int 4 \cos x dx$$

$$\ln(y^4 + 1) = 4 \sin x + c$$

or

$$\int \frac{y^3}{y^4 + 1} dy = \int \cos x dx$$

$$u = y^4 + 1$$



$$(c) \underbrace{\left(\frac{y}{x} + 6x\right)}_M + \frac{dy}{dx} \underbrace{(\ln x - 2)}_N = 0$$

$$N_x = M_y \Rightarrow \text{Exact}$$

$$\phi = y \ln x + 3x^2 - 2y = c$$

$$(d) y' + 5y = e^{-x} \sin x$$

end goal: $\left(uy\right)' = uy' + u'y \rightarrow u' = 5u$

$$u = e^{5x}$$

$$\left(e^{5x}y\right)' = e^{5x}e^{-x}\sin x$$

$$e^{5x}y = \frac{1}{15}e^{4x}(4\sin x - \cos x)$$

$$y = \frac{1}{15e^x}(4\sin x - \cos x)$$

$$\int e^{4x} \sin x dx = \frac{1}{4}e^{4x}\sin x - \int \frac{1}{4}e^{4x}\cos x dx$$

$$= \frac{1}{4}e^{4x}\sin x - \frac{1}{16}e^{4x}\cos x + \int \frac{1}{16}e^{4x}\sin x dx$$

$$\int e^{4x} \sin x dx = \frac{1}{15}e^{4x}(4\sin x - \cos x)$$

2. Estimate $y(3)$ using Euler's method with a stepsize of $\Delta t = 1$ for the differential equation

$$\frac{dy}{dt} = 3t - y, \quad y(0) = -2.$$

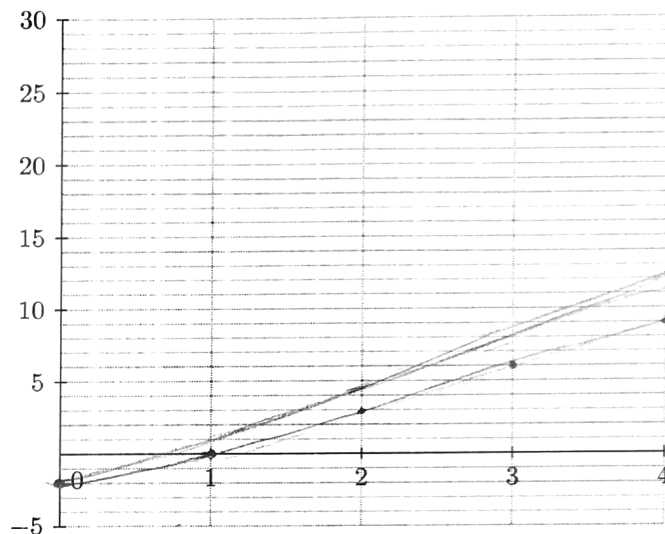
- (a) Plot your solution on the graph below.

$$\left. \frac{dy}{dt} \right|_{(0, -2)} = 2$$

$$\left. \frac{dy}{dt} \right|_{(1, 0)} = 3$$

$$\left. \frac{dy}{dt} \right|_{(2, 3)} = 3$$

$$\left. \frac{dy}{dt} \right|_{(3, 6)} = 3$$



- (b) Find the solution $y(t)$ to the initial value problem and plot it on the graph as well. Does Euler's method give an overestimate or underestimate?

$$y + y' = 3t$$

$$(e^t y)' = 3te^t$$

$$e^t y = e^t(3t - 3) + c$$

$$y = 3(t - 1) + e^{-t}$$

$$e^0(-2) = e^0(0-3) + c$$

$$c = -2 + 3 = 1$$

Underestimate

$$y'' = 3 - y' = 3 - (3t - y) = y - 3t + 3 > 0$$

3. Find the solutions to the following initial value problems:

(a) $y'' + y' = 1 + 2e^{-x}$, $y(0) = y'(0) = 1$

$$y_h'' + y_h' + 0y_h = 0$$

$$y_h = c_1 + c_2 e^{-x}$$

$$x(1-x)$$

$$y_p = c_1 x + c_2 x e^{-x}$$

$$e^{-x}(x-1-1)$$

$$y_p'' + y_p' = c_2 e^{-x}(x-1-1) + c_1 + c_2 e^{-x}(1-x) = 1 + 2e^{-2x}$$

$$c_1 = 1$$

$$-c_2 = 2$$

$$c_2 = -2$$

$$y = c_1 + c_2 e^{-x} + x - 2x e^{-x}$$

$$y(0) = c_1 + c_2 = 1$$

$$c_2 = -2$$

$$y'(0) = -c_2 + 1 - 2 = 1$$

$$c_1 = 3$$

$$y = 3 + x - 2e^{-x}(1+x)$$

(b) $y'' + 2y' + 5y = 0$, $y(0) = 1, y'(0) = 2$

$$\lambda^2 + 2\lambda + 5 = 0$$

$$\lambda = -1 \pm \sqrt{1-5}$$

$$= -1 \pm 2i$$

$$y = e^{-x} (c_1 \cos 2x + c_2 \sin 2x)$$

$$y(0) = c_1 = 1$$

$$y'(0) = e^{-x} (-c_1 \cos 2x + 2c_2 \cos 2x) \Big|_{x=0}$$

$$= -c_1 + 2c_2 = 2$$

$$c_2 = \frac{3}{2}$$

$$y(x) = e^{-x} \left(\cos 2x + \frac{3}{2} \sin 2x \right)$$

Undetermined Coefficients Review

4. Find the form of the particular solution for the following differential equations.

(a) $y'' + 4y' + 4y = \sin x + x^2 e^{-2x}$

$$\lambda^2 + 4\lambda + 4 = 0$$

$$\lambda = -2$$

$$y_h = c_1 e^{-2x} + c_2 x e^{-2x}$$

$$y_p = A \sin x + B \cos x + C x^2 e^{-2x}$$

(b) $y''' + 4y'' + 5y' = x^3 + e^{2x} \sin x$

$$\lambda^3 + 4\lambda^2 + 5\lambda = 0$$

$$\lambda = 0, \frac{-4 \pm \sqrt{16 - 20}}{2}$$

$$-2 \pm i$$

$$y_p = Ax^4 + Bx^3 + Cx^2 + Dx + e^{2x} (E \sin x + F \cos x)$$

(c) $y'''' + 2y'' + y = \sin x + x^2 \cos x$

$$\lambda^4 + 2\lambda^2 + 1 = 0$$

$$(\lambda^2 + 1)^2 = 0$$

$$\lambda = \pm i$$

$$y_p = (A \sin x + B \cos x) (x^4 + Cx^3 + Dx^2)$$

Variation of Parameters

5. Solve the following differential equations.

★ (a) $y'' - 4y = \sinh(2x)$

$$\lambda^2 - 4 = 0$$

$$\lambda = \pm 2$$

$$y_h = c_1 \sinh(2x) + c_2 \cosh(2x)$$

$$y_p = \cancel{k_1(x) \sinh(2x)} + \cancel{k_2(x) \cosh(2x)} = k_1(x) e^{2x} + k_2(x) e^{-2x}$$

$$P(k_1'' y_1 + 2k_1' y_1' + k_1 y_1'') + Q(k_2'' y_2 + k_2 y_2'') + R(k_1 y_1' + k_2 y_2') = g(x)$$

$$+ P(\quad) + Q(\quad) + R(\quad)$$

$$P(k_1' y_1 + k_2' y_2)' + k_1'(P y_1' + Q y_2') + k_2'(P y_2' + Q y_1')$$

$$\begin{cases} k_1' y_1 + k_2' y_2 = 0 \\ k_1' y_1' + k_2' y_2' = \frac{g}{P} \end{cases}$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} k_1' \\ k_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{g}{P} \end{bmatrix}$$

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \end{bmatrix} = \begin{bmatrix} 0 \\ \sinh 2x \end{bmatrix}$$

$$k_1' = \frac{y_2 \sinh(2x)}{W(y_1, y_2)} = \frac{1}{2}$$

$$k_1 = \frac{1}{8} \cosh^2 2x$$

$$k_2' = \frac{y_1 \sinh(2x)}{-2} = -\frac{\sinh^2 2x}{2} = -\frac{\cosh 4x - 1}{2}$$

$$k_2 = -\frac{1}{8} \sinh 4x - 1$$

$$= -\frac{1}{8} \sinh 2x \cosh 2x + \frac{x}{8}$$

$$y_p = \frac{x}{4} \cosh(2x)$$

(b) $y'' + 4y = \sin^2 x = \frac{1 - \cos 2x}{2}$

$$\lambda^2 + 4 = 0$$

$$\lambda = \pm 2i$$

$$y_h = c_1 \sin 2x + c_2 \cos 2x$$

$$k_1' = \frac{\sin^2 x \cos 2x}{-2} \Rightarrow k_1 = -\frac{1}{32} - \frac{1}{8}x + \frac{1}{8} \sin 2x$$

$$k_2' = \frac{\sin^2 x \cdot \sin 2x}{-2} \Rightarrow k_2 = -\frac{1}{4} \sin^4 x$$

$$y_p = \sin 2x \left(-\frac{1}{32} - \frac{1}{8}x + \frac{1}{8} \sin 2x \right)$$

$$+ \cos 2x \left(-\frac{1}{4} \sin^4 x \right)$$

(\rightarrow simplification)