Review

1. Give examples of the following or explain why they don’t exist.

(a) Values of $a$ and $b$ so that the differential equation $y'' + ay' + by = 12x^2e^x$ has particular solution $y_p = x^4e^x$.

\[
b \left( x''e^x \right) + e^x \left( x'' + 4x' \right) + e^x \left( x'' + 8x' + 12x \right) = 12x^2e^x
\]
\[
a + b = -1
\]
\[
\begin{align*}
\hat{a} &= -7 \\
\hat{b} &= 1
\end{align*}
\]

(b) A first-order differential equation with solution $x^2y^2 + e^{xy} = k$.

Implicidy differentiate both sides

\[2xy^2 + 2x^2y y' + (y + xy') e^{xy} = 0\]

(c) A stepsize for Euler’s method that overestimates the solution of the initial value problem $y' = 2y$, $y(0) = 3$ at the point $x = 5$.

\[
y' = 2y \\
y'' = 2y' = 2(2y) = 4y > 0 \Rightarrow \text{concave up, underestimate}
\]
Power Series

2. Show Euler's formula,

\[ e^{i\theta} = \cos \theta + i \sin \theta, \]

by using the Taylor series for \( e^x \), \( \cos x \), and \( \sin x \).

\[
e^x = 1 + x + \frac{1}{2!} x^2 + \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \cdots + \frac{1}{n!} x^n + \cdots
\]

\[
\sin x = x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \cdots
\]

\[
\cos x = 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \cdots
\]

\[
e^{ix} = 1 + ix + \frac{1}{2!} (-1)^2 x^2 + \frac{1}{3!} (-1)^3 x^3 + \cdots + \frac{1}{n!} (-1)^n x^n + \cdots
\]

\[
= 1 + ix - \frac{1}{2!} x^2 - \frac{1}{3!} x^3 + \frac{1}{4!} x^4 + \frac{1}{5!} x^5 - \frac{1}{6!} x^6 - \cdots
\]

\[
= \left( 1 - \frac{1}{2!} x^2 + \frac{1}{4!} x^4 - \frac{1}{6!} x^6 + \cdots \cos x \right) + i \left( x - \frac{1}{3!} x^3 + \frac{1}{5!} x^5 - \cdots \sin x \right)
\]

\[= \cos x + i \sin x \]

Q.E.D.
3. The power series

\[ 1 - x + x^2 - x^3 + \ldots = \frac{1}{1 - (-x)} \]

can be thought of as a geometric series with multiplier \(-x\).

(a) For what values of the multiplier \(x\) does the series converge?

\[ x \in (-1, 1) \]

**Ratio Test (or Alternating Series Test)**

\[ a_n = (-1)^n x^n \]

\[ \lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{x^n} \right| = |x| < 1 \]

(b) The derivative of \(\ln(1 + x)\) is \(\frac{1}{1+x}\). Use the series above to derive a power series for \(\ln(1 + x)\) by integrating the series term by term.

\[
\int \frac{d}{dx} \ln (1 + x) = \int \left(1 - x + x^2 - x^3 + x^4 + \ldots \right) dx
\]

\[ \ln (1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \ldots \]

**Note**: \(\ln 1 = 0 = c\)
4. Determine a power series solution to the following linear initial value problem.

\[ y' = (x-1)^2 y, \quad y(1) = -1 \]

Write \( y \) and \( y' \) as power series:

\[
y = \sum_{n=0}^{\infty} a_n (x-1)^n
\]

\[
y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n
\]

Subtract and simplify:

\[
\sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = (x-1)^2 \left( \sum_{n=0}^{\infty} a_n (x-1)^n \right)
\]

\[
= \sum_{n=1}^{\infty} a_n (x-1)^n (x-1)^2
\]

\[
= \sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n
\]

Reindex:

\[
\sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n = \sum_{n=2}^{\infty} a_{n-2} (x-1)^n
\]

\[
a_1 = 2 a_2 (x-1) + \sum_{n=2}^{\infty} (n+1) a_{n+1} (x-1)^n = \sum_{n=2}^{\infty} a_{n-2} (x-1)^n
\]

\[
\begin{align*}
a_1 &= 2 a_2 - 1 \\
a_2 &= 0 \\
a_0 &= y(1) = -1
\end{align*}
\]

From:

\[
y' = (x-1)^2 y \\
y'' = 2 (x-1)^2 y + (x-1)^2 y'
\]

\[
y = -1 - \frac{1}{3} (x-1)^3 - \frac{1}{18} (x-1)^6 - \cdots - \frac{1}{n! 3^n} (x-1)^{3n} - \cdots
\]

By inspection.

No good general method about the first few terms.

As engineers we only care about the first few terms.