Review

1. Give examples of the following or explain why they don't exist.

   (a) Values of a and b so that the differential equation $y'' + ay' + by = 12x^2e^x$ has particular solution $y_P = x^4e^x$.

   $$\underset{y}{b(x^4e^x)} + \underset{y'}{ae^x(x^4+4x^3)} + \underset{y''}{e^x(x^4+8x^3+\cancel{12x^2})} = 12\cancel{x^2e^x}$$

   $a + b = -1$

   $$\boxed{\begin{aligned} a &= -2 \\ b &= 1 \end{aligned}}$$

   (b) A first-order differential equation with solution $x^2y^2 + e^{xy} = k$.

   Implicitly differentiate both sides

   $$\Rightarrow \boxed{2xy^2 + 2x^2yy' + (y + xy')e^{xy} = 0}$$

   (c) A stepsize for Euler's method that overestimates the solution of the initial value problem $y' = 2y$, $y(0) = 3$ at the point $x = 5$.

   $y' = 2y$

   $y'' = 2y' = 2(2y) = 4y > 0 \Rightarrow$ concave up

   $\Downarrow$

   underestimate

Power Series

2. Show Euler's formula,

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

by using the Taylor series for e^x, $\cos x$, and $\sin x$.

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4 + \dots + \frac{1}{n!}x^n + \dots$$

$$\sin x = x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \dots$$

$$\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots$$

$$e^{ix} = 1 + ix + \frac{1}{2}i^2x^2 + \frac{1}{6}i^3x^3 + \frac{1}{24}i^4x^4 + \dots$$

$$= 1 + ix - \frac{1}{2}x^2 - \frac{1}{6}ix^3 + \frac{1}{24}x^4 + \frac{1}{120}ix^5 - \frac{1}{720}x^6 - \dots$$

$$(+1 \quad +i \quad -1 \quad -i \quad +1 \quad +i \quad -1 \quad -i)$$

$$= 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \frac{1}{720}x^6 + \dots \quad \cos x$$

$$+ ix - \frac{1}{6}ix^3 + \frac{1}{120}ix^5 - \dots$$

$$= \cos x$$

$$+ i\left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \dots\right)$$

$$= \cos x + i\sin x$$

Q.E.D.

3. The power series

$$1 - x + x^2 - x^3 + \cdots = \frac{1}{1-(-x)}$$

can be thought of as a geometric series with multiplier $-x$.

(a) For what values of the multiplier x does the series converge?

$x \in (-1, 1)$

Ratio test (or Alternating series test)

$a_n = (-1)^n x^n$

$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{x^{n+1}}{x^{n}} \right| = \boxed{|x| < 1}$

(b) The derivative of $\ln(1+x)$ is $\frac{1}{1+x}$. Use the series above to derive a power series for $\ln(1+x)$ by integrating the series term by term.

$\int \frac{d}{dx} \ln(1+x) = \int 1 - x + x^2 - x^3 + x^4 - \ldots$

$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} + \ldots$

Note: $\ln 1 = 0 = c$

4. Determine a power series solution to the following linear initial value problem.

$$y' = (x-1)^2 y, \quad y(1) = -1$$

center @ 1

Write y & y' as power series

$$y = \sum_{n=0}^{\infty} a_n (x-1)^n$$

$$y' = \sum_{n=1}^{\infty} n a_n (x-1)^{n-1} \quad \left(= \sum_{n=0}^{\infty} n a_n (x-1)^{n-1} \right)$$

Substitute & simplify

$$y' = (x-1)^2 y$$

$$\sum_{n=1}^{\infty} n a_n (x-1)^{n-1} = (x-1)^2 \left(\sum_{n=0}^{\infty} a_n (x-1)^n \right)$$

$$= \sum \left(a_n (x-1)^n (x-1)^2 \right)$$

$$= \sum_{n=0}^{\infty} a_n (x-1)^{n+2}$$

Reindex

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} (x-1)^n = \sum_{n=2}^{\infty} a_{n-2} (x-1)^n$$

$$a_1 + 2a_2(x-1) + \sum_{n=2}^{\infty} (n+1) a_{n+1} (x-1)^n = \sum_{n=2}^{\infty} a_{n-2} (x-1)^n$$

* $\{(x-1)^0, (x-1)^1, (x-1)^2, \ldots (x-1)$ are linearly independent

$a_1 = 0$

$a_2 = 0$

$a_0 = y(1) = -1$

or

$y(1) = a_0 = -1$

$y'(1) = 1 \cdot a_1 = 0$

$y''(1) = 2a_2 = 0$

from

$y' = (x-1)^2 y$

$y'' = 2(x-1)y + (x-1)^2 y'$

$(n+1) a_{n+1} = a_{n-2}$ for $n \geq 2$

$n a_n = a_{n-3}$ for $n \geq 3$

$a_n = \dfrac{a_{n-3}}{n}$ for $n \geq 3$

$$y = -1 - \frac{1}{3}(x-1)^3 - \frac{1}{18}(x-1)^6 - \cdots - \frac{1}{m!\,3^m}(x-1)^{3m} - \cdots$$

by "inspection"

No good general method $\Rightarrow$ As engineers, we only care about the first few terms