Analytic Solutions

1. For the IVP

\[(1 - x)y'' + xy' - y = 0, \quad y(0) = -3, y'(0) = 2\]

(a) Determine the minimum radius of convergence of solutions around \(x_0 = 0\).

(b) If \(y = \phi(x)\) is a solution of the IVP, find \(\phi''(0), \phi'''(0),\) and \(\phi''''(0)\).

(c) Write down the first five terms of the analytic power series solution

\[y = \sum_{n=0}^{\infty} a_n x^n\]

by using the relationship \(n!a_n = \phi^{(n)}(x_0)\).
More Power Series

2. Find a general solution to the following differential equation using the power series method.

\[ y'' + xy = 0, \quad y(0) = 1, y'(0) = 0. \]
Euler Equations

3. Solve the Euler equation $2x^2y'' + 3xy' - y = 0, \ x > 0$ by looking for solutions of the form $y = x^r$.

(a) Use the Wronskian to show that the two solutions are linearly independent for $x > 0$. 
4. For the Euler equation,

\[ x^2 y'' + 5xy' + 4y = 0, \quad x > 0, \]

(a) Find one solution \( y_1(x) \) by making the substitution \( y = x^r \).

(b) Use the method of reduction of order to find the other solution:

i. Assume a second solution of the form \( y_2(x) = u(x)y_1(x) \). Plug into the differential equation and simplify to an equation involving \( u'' \) and \( u' \).

ii. Solve for \( u' \),

iii. Antidifferentiate to determine \( u \).