Analytic Solutions

1. For the IVP

$$\overset{P}{(1-x)}y'' + \overset{Q}{x}y' - \overset{R}{y} = 0, \quad y(0) = -3, y'(0) = 2$$

(a) Determine the minimum radius of convergence of solutions arond $x_0 = 0$.

$P = 1-x \Rightarrow P(1) = 0$

$\Downarrow$

$RoC = (1)-(0) = \boxed{1}$

(b) If $y = \phi(x)$ is a solution of the IVP, find $\phi''(0), \phi'''(0)$, and $\phi''''(0)$.

$\phi(0) = -3$

$\phi'(0) = 2$

$(1-x)y'' + xy' - y = 0$

$y'' = \frac{-x}{1-x}y' + \frac{1}{1-x}y$

$\phi''(0) = 0\phi'(0) + 1\phi(0)$
$= \boxed{-3}$

$(1-x)y''' - 1y'' + xy'' + \cancel{1y'} - \cancel{y'} = 0$

$\phi'''(0) = 1 \cdot \phi''(0) = \boxed{-3}$

$(1-x)y^{(4)} + (-1-1+x)y''' + 1y'' = 0$

$\phi^{(4)}(0) = 2\phi'''(0) - 1\phi''(0)$

$= -6 + 3 = \boxed{-3}$

(c) Write down the first five terms of the analytic power series solution

$$y = \sum_{n=0}^{\infty} a_n x^n$$

by using the relationship $n!a_n = \phi^{(n)}(x_0)$.

$a_0 = y(0) = -3$

$a_1 = y'(0) = 2$

$a_2 = \frac{1}{2}y''(0) = -\frac{3}{2}$

$a_3 = \frac{1}{6}y'''(0) = -\frac{1}{2}$

$a_4 = \frac{1}{24}y^{(4)}(0) = -\frac{1}{8}$

$\boxed{y(x) = -3 + 2x - \frac{3}{2}x^2 - \frac{1}{2}x^3 - \frac{1}{8}x^4 + \ldots}$

More Power Series

2. Find a general solution to the following differential equation using the power series method.

$$y'' + xy = 0, \quad y(0) = 1, y'(0) = 0.$$

$\downarrow$

$x_0 = 0$

$$\boxed{y = \sum_{n=0}^{\infty} a_n x^n}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$$

$$y'' + xy = 0$$

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$(2)(1) a_2 x^0 + \sum_{n=1}^{\infty} \left((n+2)(n+1) a_{n+2} + a_{n-1} \right) x^n = 0$$

$\downarrow$

$a_2 = 0$

$$(n+2)(n+1) a_{n+2} + a_{n-1} = 0 \quad \text{for } n \geq 1$$

$$(n+3)(n+2) a_{n+3} + a_n = 0 \quad \text{for } n \geq 0$$

$$\boxed{a_{n+3} = -\frac{a_n}{(n+2)(n+3)}} \quad \text{for } n \geq 0$$

$$\boxed{\begin{aligned} a_0 &= y(0) = 1 \\ a_1 &= y'(0) = 0 \\ a_2 &= 0 \end{aligned}}$$

$$\Rightarrow \boxed{y = 1 - \frac{1}{6} x^3 + \frac{1}{180} x^6 \cdots + \frac{(-1)^n}{2 \cdot 3 \cdot 5 \cdot 6 \cdots 3n} x^{3n} + \cdots}$$

$$a_3 = -\frac{1}{2 \cdot 3} = -\frac{1}{6}$$

$$a_6 = +\frac{1/6}{5 \cdot 6} = \frac{1}{180}$$

$$a_{3n} = \frac{(-1)^n}{2 \cdot 3 \cdot 5 \cdot 6 \cdot \ldots \cdot (3n-1)(3n)}$$

Euler Equations

3. Solve the Euler equation $2x^2y'' + 3xy' - y = 0,\ x > 0$ by looking for solutions of the form $y = x^r$.

$y = x^r$

$y' = rx^{r-1}$

$y'' = r(r-1)x^{r-2}$

$2x^2 r(r-1)x^{r-2} + 3xrx^{r-1} - x^r = 0$

$2r(r-1)x^r + 3rx^r - x^r = 0$

$(2r(r-1) + 3r - 1)\cancel{x^r} \overset{x>0}{=} 0$

$2r^2 - 2r + 3r - 1 = 0$

$2r^2 + r - 1 = 0$

$(2r-1)(r+1) = 0$

$r = \frac{1}{2}, -1$

$\Downarrow$

$$\boxed{y = c_1\sqrt{x} + \frac{c_2}{x}}$$

(a) Use the Wronskian to show that the two solutions are linearly independent for $x > 0$.

$$W\left(\sqrt{x}, \frac{1}{x}\right) = \begin{pmatrix} \sqrt{x} & \frac{1}{x} \\ \frac{1}{2\sqrt{x}} & -\frac{1}{x^2} \end{pmatrix} = -x^{-3/2} - \frac{1}{2}x^{-3/2}$$

$$= -\frac{3}{2}x^{-3/2} \neq 0 \quad \forall x > 0$$

$\Downarrow$

l. i.

4. For the Euler equation,

$$x^2y'' + 5xy' + 4y = 0, \quad x > 0,$$

(a) Find one solution $y_1(x)$ by making the substitution $y = x^r$.

$y = x^r$

$y' = rx^{r-1}$

$y'' = r(r-1)x^{r-2}$

$x^2 r(r-1)x^{r-2} + 5x\, r x^{r-1} + 4x^r = 0$

$r(r-1) + 5r + 4 = 0$

$r = -2 \Rightarrow y_1 = \frac{1}{x^2}$

(b) Use the method of reduction of order to find the other solution:

i. Assume a second solution of the form $y_2(x) = u(x)y_1(x)$. Plug into the differential equation and simplify to an equation involving u'' and u'.

ii. Solve for u',

iii. Antidifferentiate to determine u.

$y_2 = u y_1$

$y_2' = u'y_1 + uy_1'$

$y_2'' = u''y_1 + 2u'y_1' + uy_1''$

$x^2(u''y_1 + 2u'y_1' + uy_1'') + 5x(u'y_1 + uy_1') + 4uy_1 = 0$

$\underbrace{x^2y_1}_{1} u'' + \underbrace{(2x^2y_1' + 5xy_1)}_{-\frac{4}{x} + \frac{5}{x}} u' + \overbrace{(x^2y_1'' + 5xy_1' + 4y_1)}^{0} u = 0$

$u'' + \frac{1}{x}u' = 0$

$(xu')' = 0 \cdot x = 0$

$xu' = \cancel{x}$

$u = \int \frac{1}{x}\,dx = \ln x + \cancel{c}$

$y_2 = (\ln x) \cdot \frac{1}{x^2}$

$$\boxed{y = \frac{c_1}{x^2} + \frac{c_2 \ln x}{x^2}}$$