

Laplace Transforms

1. To solve the problem $y'' + 3y' + 2y = e^t$, $y(0) = 1$, $y'(0) = -1$.

(a) First find the Laplace transform of the entire equation.

$$\mathcal{L}\{y'' + 3y' + 2y\} = \mathcal{L}\{e^t\}$$

$$\begin{aligned} s^2 Y - sy(0) - y'(0) \\ + 3sY - 3y(0) \\ + 2Y \end{aligned} = \frac{1}{s-1}$$

(b) Then use the initial conditions and solve for $\mathcal{L}(y)$.

$$(s^2 + 3s + 2) Y = \frac{1}{s-1} + (s+2)$$

$$Y = \frac{1}{(s-1)(s+2)(s+1)} + \frac{1}{s+1}$$

(c) Then find the partial fractions of the right side of your equation and use the Laplace table to find y .

$$Y = \frac{1}{(s+1)(s+2)(s-1)} + \frac{1}{s+1}$$

$$= \frac{1/3}{s+2} + \frac{-1/2}{s+1} + \frac{1/6}{s-1} + \frac{1}{s+1}$$

$$\lambda^2 + 3\lambda + 2 = 0$$

$$(\lambda+2)(\lambda+1) = 0$$

$$\lambda = -1, -2$$

↓

makes sense

$$y(t) = \frac{1}{3}e^{-2t} + \frac{1}{6}e^t + \frac{1}{2}e^{-t}$$

2. Solve the following differential equations using Laplace transforms.

(a) $y'' + 3y' + 2y = 3\sin 2t, y(0) = 1, y'(0) = 0.$

$$s^2 Y - s + 3(sY - 1) + 2Y = \frac{6}{s^2 + 4}$$

$$(s^2 + 3s + 2) Y = \frac{6}{s^2 + 4} + s + 3$$

$$Y = \frac{6}{(s^2 + 4)(s+1)(s+2)} + \frac{s+3}{(s+2)(s+1)}$$

$$= \frac{6/s}{s+1} + \frac{-3/4}{s+2} + \frac{-9s-6}{20(s^2+4)} + \frac{2}{s+1} + \frac{-1}{s+2}$$

$$y = \frac{16}{5} e^{-t} - \frac{7}{4} e^{-2t} - \frac{9}{20} \cos 2t - \frac{3}{20} \sin 2t$$

(b) $y'' + 3y' + 2y = 3u(t-2), y(0) = 0, y'(0) = 2.$

$$s^2 Y - 2 + 3sY + 2Y = \frac{3e^{-2s}}{s}$$

$$(s^2 + 3s + 2) Y = \frac{3e^{-2s}}{s} + 2$$

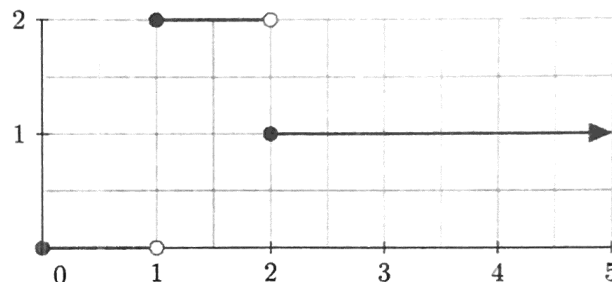
$$Y = e^{-2s} \frac{3}{s(s+1)(s+2)} + \frac{2}{(s+1)(s+2)}$$

$$= e^{-2s} \left(\frac{3/2}{s} + \frac{-3}{s+1} + \frac{3/2}{s+2} \right) + \frac{1}{s+1} + \frac{-1}{s+2}$$

$$= u(t-2) \left[\frac{3}{2} - 3e^{-t} + \frac{3}{2} e^{-2t} \right] + e^{-t} - e^{-2t}$$

3. Solve the following initial value problem

$$y'' + 2y' + 5y = \begin{cases} 0 & 0 \leq t < 1 \\ 2 & 1 \leq t < 2 \\ 1 & 2 \leq t \end{cases}, \quad y(0) = 1, y'(0) = -1.$$



- (a) First show that the right hand side, which is shown on the graph above, can be written as $2u(t-1) - u(t-2)$.

Inspection

- (b) Then use Laplace transforms to solve the initial value problem.

$$\begin{aligned} s^2 Y - s + 1 \\ + 2(sY - 1) \\ + 5Y \end{aligned} = \frac{2e^{-t}}{s} - \frac{e^{-2t}}{s}$$

$$(s^2 + 2s + 5) Y = \frac{2e^{-t}}{s} - \frac{e^{-2t}}{s} + s + 1$$

$$Y = e^{-t} \frac{2}{s(s^2 + 2s + 5)} + e^{-2t} \frac{1}{s(s^2 + 2s + 5)} + \frac{s+1}{s^2 + 2s + 5}$$

$$= \left(\frac{1/5}{s} - \frac{1}{10} \frac{2s+4}{s^2 + 2s + 5} \right) (2e^{-t} + e^{-2t}) + \frac{s+1}{s^2 + 2s + 5}$$

$(s+1)^2 + 4$

$$y = \left[\frac{1}{5} - \frac{1}{10} (2e^{-t} \cos 2t + 1e^{-t} \sin 2t) \right] (2u(t-1) + u(t-2)) + e^{-t} \cos 2t$$