Laplace Transform

1. Solve the following initial value problems using the Laplace Transform

(a) $y'' + 3y' + 2y = \sin x, \quad y(0) = 1, y'(0) = 2$

$$s^2 Y - s y(0) - y'(0) + 3sY - 3y(0) + 2Y = \frac{1}{s^2+1}$$

$$(s^2+3s+2)Y = \frac{1}{s^2+1} + s + 5$$

$$Y = \frac{1}{(s^2+1)(s+2)(s+1)} + \frac{s+5}{(s+2)(s+1)} = \frac{-\frac{3}{10}s+\frac{1}{10}}{s^2+1} + \frac{-1/5}{s+2} + \frac{1/2}{s+1} + \frac{-3}{s+2} + \frac{4}{s+1}$$

$$\boxed{y(x) = -\frac{3}{10}\cos t + \frac{1}{10}\sin t - \frac{16}{5}e^{-2t} + \frac{9}{2}e^{-t}}$$

(b) $y'' + 3y' + 2y = x^2 e^{-x}, \quad y(0) = 3, y'(0) = -1$

$$(s^2+3s+2)Y = \frac{2}{(s+1)^3} + 3s + 8$$

$$Y = \frac{2}{(s+1)^3(s+2)(s+1)} + \frac{3s+8}{(s+2)(s+1)}$$

$$= \frac{2}{(s+1)^4} - \frac{2}{(s+1)^3} + \frac{2}{(s+1)^2} + \frac{3}{s+1}$$

$$\boxed{y(x) = \left(\frac{1}{3}t^3 - t^2 + 2t + 3\right)e^{-t}}$$

2. Find $\mathcal{L}^{-1}\left\{\frac{1}{(s-4)(s+1)}\right\}$ by taking a convolution.

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-4)(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{t-4}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{t+1}\right\}$$

$$= e^{+4t} * e^{-t}$$

$$= \int_0^t e^{4\tau} e^{-t+\tau} \, d\tau$$

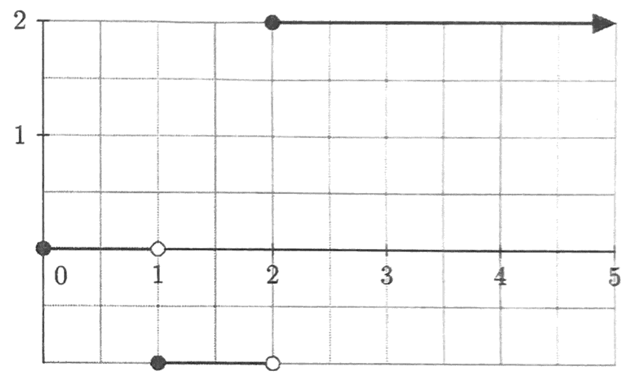
$$= \int_0^t e^{-t+5\tau} \, d\tau$$

$$= \frac{1}{5} e^{-t+5\tau} \Big|_0^t$$

$$= \boxed{\frac{1}{5}\left(e^{4t} - e^{-t}\right)}$$

3. Solve the following initial value problem (piece-wise function shown below).

$$y'' + 2y' + y = \begin{cases} 0 & 0 \le t < 1 \\ -1 & 1 \le t < 2 \\ 2 & 2 \le t \end{cases}, \quad y(0) = 1, y'(0) = -1.$$



$g(t) = -u(t-1) + 3u(t-2)$

$s^2Y - sy(0) - y'(0)$

$+2sY - 2y(0) \quad = -\frac{e^{-s}}{s} + \frac{3e^{-2s}}{s}$

$+Y$

$(s^2 + 2s + 1)Y = \left(\frac{3e^{-2s}}{s} - \frac{e^{-s}}{s}\right) + s + 1$

$Y = (3e^{-2s} - e^{-s})\frac{1}{s(s+1)^2} + \frac{1}{s+1} = (3e^{-2s} - e^{-s})\left(\frac{1}{s} - \frac{1}{(s+1)^2} - \frac{1}{s+1}\right) + \frac{1}{s+1}$

$y(t) = (1 - te^{-t} - e^{-t}) * (3\delta(t-2) - \delta(t-1)) + e^{-t}$

$= (te^{-t}) * (3\delta(t-2) - \delta(t-1)) + e^{-t}$

4. Solve the following initial value problem

$$y'' - 3y' - 4y = \delta(t-1) - 2\delta(t-2), \quad y(0) = 2, y'(0) = -1$$

$$(s^2 - 3s - 4)Y = e^{-s} - 2e^{-2s} - 2s - 1$$

$$Y = (e^{-s} - 2e^{-2s})\frac{1}{(s-4)(s+1)} - \frac{2s+1}{(s-4)(s+1)}$$

$$= (e^{-s} - 2e^{-2s})\left(\frac{1/5}{s-4} - \frac{1/5}{s+1}\right) + \frac{1/5}{s-4} \quad \frac{9/5}{s+1}$$

$$\boxed{\begin{aligned} y(t) &= \left(\tfrac{1}{5}e^{+4t} - \tfrac{1}{5}e^{-t}\right) * \left(\delta(t-1) - 2\delta(t-2)\right) \\ &\quad + \tfrac{1}{5}e^{4t} + \tfrac{9}{5}e^{-t} \end{aligned}}$$

$$= \frac{1}{5}e^{4t-4}u(t-1) - \frac{1}{5}e^{-t+1}u(t-1)$$

$$- \frac{2}{5}e^{4t-8}u(t-2) + \frac{2}{5}e^{-t+2}u(t-2)$$

$$+ \frac{-1}{5}e^{4t} + \frac{9}{5}e^{-t}$$