

Laplace Transform

1. Solve the following initial value problems using the Laplace Transform

(a) $y'' + 3y' + 2y = \sin x$, $y(0) = 1, y'(0) = 2$

$$\begin{aligned} s^2 Y - sy(0) - y'(0) \\ + 3sY - 3y(0) \\ + 2Y \end{aligned} = \frac{1}{s^2 + 1}$$

$$(s^2 + 3s + 2) Y = \frac{1}{s^2 + 1} + s + 5$$

$$Y = \frac{1}{(s+1)(s+2)(s+1)} + \frac{s+5}{(s+2)(s+1)} = \frac{-\frac{2}{10}s + \frac{4}{10}}{s^2 + 1} + \frac{-1/5}{s+2} + \frac{1/2}{s+1} + \frac{-3}{s+2} + \frac{4}{s+1}$$

$$y(x) = -\frac{3}{10} \cos t + \frac{1}{10} \sin t - \frac{16}{5} e^{-2t} + \frac{9}{2} e^{-t}$$

(b) $y'' + 3y' + 2y = x^2 e^{-x}$, $y(0) = 3, y'(0) = -1$

$$(s^2 + 3s + 2) Y = \frac{2}{(s+1)^3} + 3s + 8$$

$$Y = \frac{2}{(s+1)^3 (s+2)(s+1)} + \frac{3s+8}{(s+2)(s+1)}$$

$$= \frac{2}{(s+1)^4} - \frac{2}{(s+1)^3} + \frac{2}{(s+1)^2} + \frac{3}{s+1}$$

$$y(x) = \left(\frac{1}{3} t^3 - t^2 + 2t + 3 \right) e^{-t}$$

2. Find $\mathcal{L}^{-1}\left\{\frac{1}{(s-4)(s+1)}\right\}$ by taking a convolution.

$$\mathcal{L}^{-1}\left\{\frac{1}{(s-4)(s+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s-4}\right\} * \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= e^{4t} * e^{-t}$$

$$= \int_0^t e^{4\tau} e^{-t+\tau} d\tau$$

$$= \int_0^t e^{-t+5\tau} d\tau$$

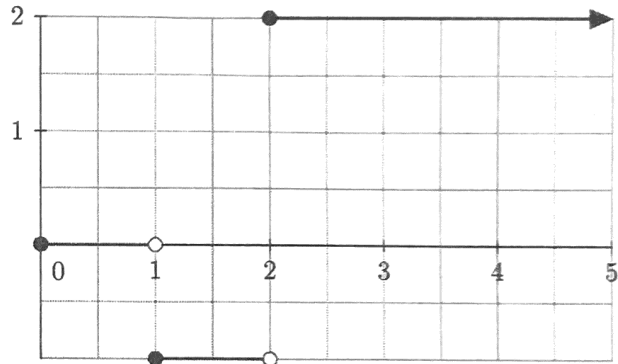
$$= \frac{1}{5} e^{-t+5\tau} \Big|_0^t$$

$$= \frac{1}{5} (e^{4t} - e^{-t})$$

3. Solve the following initial value problem (piece-wise function shown below).

$$y'' + 2y' + y = \begin{cases} 0 & 0 \leq t < 1 \\ -1 & 1 \leq t < 2 \\ 2 & 2 \leq t \end{cases}, \quad y(0) = 1, y'(0) = -1.$$

$$g(t) = -u(t-1) + 3u(t-2)$$



$$\begin{aligned} s^2 Y - sy(0) - y'(0) \\ + 2sY - 2y(0) \\ + Y \end{aligned} = -\frac{e^{-s}}{s} + 3\frac{e^{-2s}}{s}$$

$$(s^2 + 2s + 1)Y = \left(\frac{3e^{-2s}}{s} - \frac{e^{-s}}{s}\right) + s + 1$$

$$Y = (3e^{-2s} - e^{-s}) \frac{1}{s(s+1)^2} + \frac{1}{s+1} = (3e^{-2s} - e^{-s}) \left(\frac{1}{s} - \frac{1}{(s+1)^2} - \frac{1}{s+1}\right) + \frac{1}{s+1}$$

$$y(t) = (1 - te^{-t} - e^{-t}) * (3\delta(t-2) - \delta(t-1)) + e^{-t}$$

$$= (te^{-t}) * (3\delta(t-2) - \delta(t-1)) + e^{-t}$$

4. Solve the following initial value problem

$$y'' - 3y' - 4y = \delta(t-1) - 2\delta(t-2), \quad y(0) = 2, y'(0) = -1$$

$$(s^2 - 3s - 4)Y = e^{-s} - 2e^{-2s} - 2s - 1$$

$$Y = (e^{-s} - 2e^{-2s}) \frac{1}{(s-4)(s+1)} - \frac{2s+1}{(s-4)(s+1)}$$

$$= (e^{-s} - 2e^{-2s}) \left(\frac{1/5}{s-4} - \frac{1/5}{s+1} \right) + \frac{1/5}{s-4} - \frac{9/5}{s+1}$$

$$y(t) = \left(\frac{1}{5} e^{4t} - \frac{1}{5} e^{-t} \right) * \left(\delta(t-1) - 2\delta(t-2) \right) + \frac{1}{5} e^{4t} + \frac{9}{5} e^{-t}$$

$$= \frac{1}{5} e^{4t-4} u(t-1) - \frac{1}{5} e^{-t+1} u(t-1)$$

$$- \frac{2}{5} e^{4t-8} u(t-2) + \frac{2}{5} e^{-t+2} u(t-2)$$

$$+ \frac{1}{5} e^{4t} + \frac{9}{5} e^{-t}$$