Change of Variable

1. Find the area between the ellipses $x^2+4y^2 = 4$ and $x^2+4y^2 = 16$. [Hint: the change of variables $\underbrace{u = x, v = 2y}$ might be helpful.]

$$u^2 + v^2 = 4$$
$$u^2 + v^2 = 16$$

$\Rightarrow$ let $u = r\cos\theta$, $v = r\sin\theta$

$$\frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\frac{\partial(u,v)}{\partial(x,y)}} = \frac{1}{\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix}} = \frac{1}{2}$$

$$A = \int_0^{2\pi}\int_2^4 \frac{\partial(x,y)}{\partial(u,v)} \frac{\partial(u,v)}{\partial(r,\theta)} \, dr\, d\theta$$

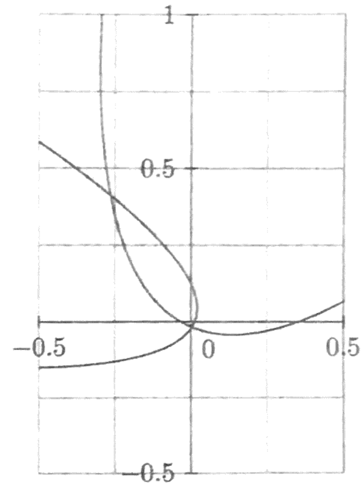
$$= \int_0^{2\pi}\int_2^4 \frac{1}{2} r \, dr\, d\theta$$

$$= \boxed{16\pi}$$

2. Calculate the area in the plane bounded by the rotated parabolas $y-2x=(3y+x)^2$ and $(3y+x)=(y-2x)^2$ shown below.

Let $u = y-2x$

$v = 3y+x$



$\Rightarrow \quad u = v^2$

$v = u^2$

$$A = \int_0^1 \int_{u^2}^{\sqrt{u}} \left| \frac{\partial(x,y)}{\partial(u,v)} \right| dv\,du$$

$$= \int_0^1 \int_{u^2}^{\sqrt{u}} \frac{1}{7}\, dv\,du \qquad \frac{\partial(x,y)}{\partial(u,v)} = \frac{1}{\begin{vmatrix} -2 & 1 \\ 1 & 3 \end{vmatrix}}$$

$$= \frac{1}{7}\left[\frac{2u^{3/2}}{3} - \frac{u^3}{3}\right]_0^1$$

$$= \frac{1}{7}\left(\frac{2}{3} - \frac{1}{3}\right) = \boxed{\frac{1}{21}}$$

3. Let T be the solid bounded by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

If the density is given by $\delta(x, y, z) = z^2$ find M, the mass of the solid.

Let $u = \frac{x}{a}$

$v = \frac{y}{b}$ $\qquad \delta = c^2 w^2$ $\qquad \frac{\partial(x,y,z)}{\partial(u,v,w)} = abc$

$w = \frac{z}{c}$

$$M = \int_{-1}^{1} \int_{-\sqrt{1-u^2}}^{\sqrt{1-u^2}} \int_{-\sqrt{1-u^2-v^2}}^{\sqrt{1-u^2-v^2}} abc\,c^2 w^2 \, dw\,dv\,du$$

$$= \int_0^{\pi} \int_0^{2\pi} \int_0^1 abc\,c^2 \rho^2 \cos^2\phi \cdot \rho^2 \sin\phi \, d\rho\,d\theta\,d\phi$$

$$= 2\pi abc^3 \int_0^{\pi} \frac{1}{5} \cos^2\phi \sin\phi \, d\phi$$

$$= 2\pi abc^3 \left[-\frac{1}{15} \cos^3\phi \right]_0^{\pi}$$

$$= 2\pi abc^3 \left(\frac{1}{15} + \frac{1}{15} \right)$$

$$= \boxed{\frac{4\pi abc^3}{15}}$$

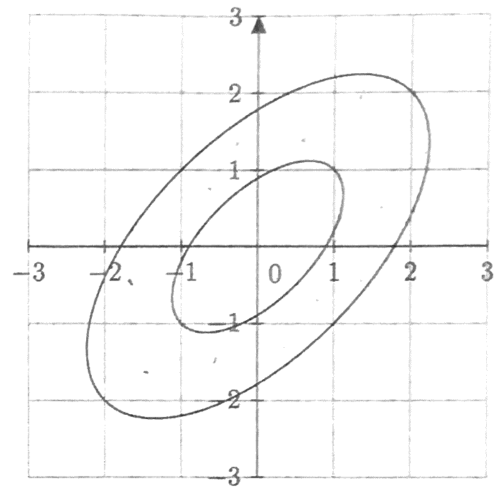
4. Find the area between the rotated ellipses $5x^2-6xy+5y^2=4$ and $5x^2-6xy+5y^2=16$. [Hint: try the substitutions $x=u+v, y=u-v$.]

$$5x^2-6xy+5y^2=4$$

$$5(u+v)^2-6(u+v)(u-v)+5(u-v)^2=4$$

$$4u^2+16v^2=4$$

$$\begin{cases} u^2+4v^2=1 \\ u^2+4v^2=4 \end{cases}$$



$$A=\left|\frac{\partial(x,y)}{\partial(u,v)}\right|\cdot\left(\text{Area in } u,v \text{ space}\right)$$

$$=2\left(2\pi-\frac{1}{2}\pi\right)$$

$$=\boxed{3\pi}$$