# Dot Product with a unit vector

1. The dot product of two vectors $\mathbf{x}$ and $\mathbf{y}$ can be defined as

$$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}| \cos\theta,$$

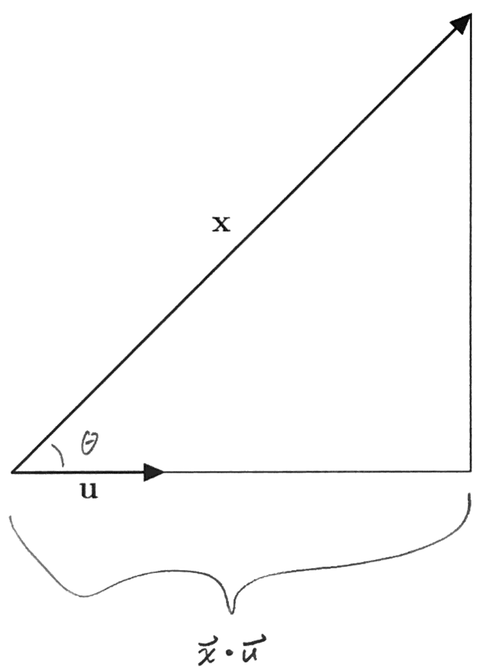
where $\theta$ is the angle between $\mathbf{x}$ and $\mathbf{y}$.

(a) Show that if $\mathbf{y} = \mathbf{u}$ is a unit vector, $\mathbf{x} \cdot \mathbf{u} = |\mathbf{x}| \cos\theta$.

$$\vec{x} \cdot \vec{u} = |\vec{x}|\cancel{|\vec{u}|}^{1} \cos\theta$$

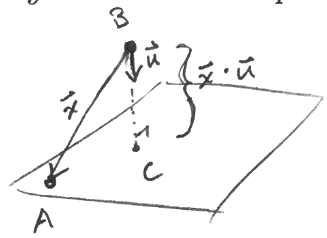
$$= |\vec{x}| \cos\theta$$

(b) Draw this length on the figure below and describe the length.




is the component of $\vec{x}$ in the direction of $\vec{u}$

2. Use the idea from the previous page to find the shortest distance from the plane $x + 2y - 2z = 6$ to the point (-2, -2, -1).




Choose some arbitrary point A on the plane

$A\ (6, 0, 0)$

$$\vec{x} = \vec{A} - \vec{B} = \langle 8, 2, 1 \rangle$$

$$\vec{u} = \hat{n} = \frac{\langle 1, 2, -2 \rangle}{|\langle 1, 2, -2 \rangle|} = \frac{1}{3}\langle 1, 2, -2 \rangle$$

$$d = |\vec{x} \cdot \vec{u}| = \left|\frac{1}{3}(8 + 4 - 2)\right|$$

$$= \boxed{\frac{10}{3}}$$

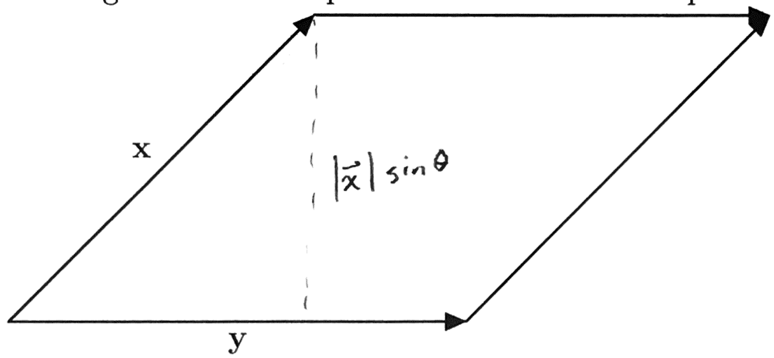
# Cross Product

3. The cross product, unlike the dot product, is a vector, which has a length and a direction. The length of the cross product of two vectors $\mathbf{x}$ and $\mathbf{y}$ can be defined as

$$|\mathbf{x} \times \mathbf{y}| = |\mathbf{x}||\mathbf{y}| \sin\theta,$$

where $\theta$ is the angle between $\mathbf{x}$ and $\mathbf{y}$. The direction of the cross product is perpendicular to both $\mathbf{x}$ and $\mathbf{y}$.

(a) Given the vectors in the figure below, label the length $|\mathbf{x}| \sin\theta$. Explain why the length of the cross product is the area of the parallelogram.




$$\text{Area} = b \cdot h$$
$$= |\vec{y}| \cdot |\vec{x}| \sin\theta$$
$$= |\vec{x}||\vec{y}| \sin\theta$$
$$= |\vec{x} \times \vec{y}|$$

(b) Does the direction of $\mathbf{x} \times \mathbf{y}$ point into the page or out of the page? What about $\mathbf{y} \times \mathbf{x}$?

out

R.H.R.

4. Use your understanding of the dot product and cross product to solve the following problems.

line of intersection is $\perp$ to both $\vec{n}$'s

(a) Find an equation for the line of intersection of the planes $x + 2y - 2z = 6$ and $2x - y + 2z = 12$.

plane 1

plane 2

$\vec{n}_1 = \langle 1, 2, -2 \rangle$

$\vec{n}_2 = \langle 2, -1, 2 \rangle$

$\vec{v} = |\vec{n}_1 \times \vec{n}_2|$

$= \langle 6, -6, -5 \rangle$

Find point of intersection:

Let $x = 0$

$\Downarrow$

$y = 18, \; z = 15$

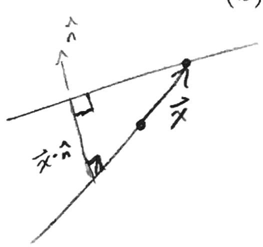
$\vec{r}_0 = \langle 0, 18, 15 \rangle$

$\vec{r} = \vec{v}t + \vec{r}_0$

$\langle x, y, z \rangle = \langle 6, -6, -5 \rangle t + \langle 0, 18, 15 \rangle$

$$\begin{cases} x = 6t \\ y = -6t + 18 \\ z = -5t + 15 \end{cases}$$

(b) Find the distance between the skew lines parameterized by $\mathbf{r}(t) = \langle 1 + 2t, 2 - 2t, 3 + t \rangle$ and $\mathbf{s}(t) = \langle -1 - 2t, -2 + 2t, -3 - t \rangle$.




$\vec{n} = \langle 2, -2, 1 \rangle \times \langle -2, 2, -1 \rangle$

$= \langle 0, 0, 0 \rangle \longrightarrow$ lines are actually parallel

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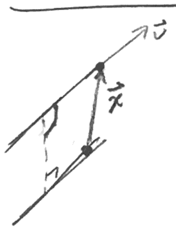



$$d = \left| \vec{x} - \frac{\vec{x} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} \right|$$

$$= \left| \langle 2, 4, 6 \rangle - \frac{2}{9} \langle 2, -2, 1 \rangle \right|$$

$$= \left| \left\langle \frac{14}{9}, \frac{40}{9}, \frac{52}{9} \right\rangle \right|$$

$$= \boxed{\frac{5}{3}\sqrt{5}}$$

$\vec{x} = \langle 1, 2, 3 \rangle - \langle -1, -2, -3 \rangle$

$= \langle 2, 4, 6 \rangle$

$\vec{v} = \langle 2, -2, 1 \rangle$

1125

$5\sqrt{45}$

$15\sqrt{5}$

# Level Sets and Graphs

5. Given the plane $x + 2y + 3z = 0$,

   (a) Find a parameterization $\mathbf{r}(x, y) : \mathbb{R}^2 \to \mathbb{R}^3$ for this plane.

$$\begin{cases} x = x \\ y = y \\ z = \frac{1}{3}(-x-2y) \end{cases}$$

6. Write down a function $g : \mathbb{R}^3 \to \mathbb{R}^1$.

$g(x, y, z) = xyz$

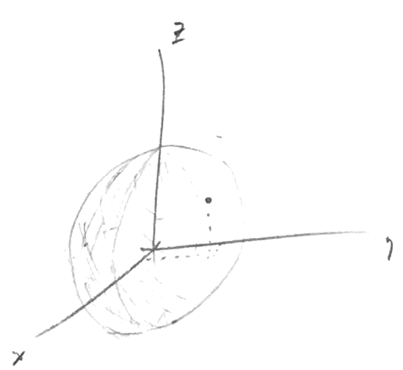
   (a) The graph of $g$ is a subset of $\mathbb{R}^4$.

   (b) A level set of $g$ is a subset of $\mathbb{R}^3$.

7. Find the level surface of the function $f(x, y, z) = x^2 + y^2 + z^2$ that goes through the point $(1, 2, 2)$. Draw a sketch of this surface.

$f(1,2,2) = 9 \Rightarrow \boxed{x^2 + y^2 + z^2 = 9}$

↳ sphere



8. Given the function $g(x,y,z) = x^2 + y^2 - z^2$.

   (a) Find the level surface that goes through the point $(3,4,5)$.

   (b) Find the level surface that goes through the point $(5,0,0)$.

   (c) Find the level surface that goes through the point $(0,0,4)$.

   (d) Draw a sketch of all three of these surfaces on the same axes.

a) $x^2 + y^2 - z^2 = 0$

b) $x^2 + y^2 - z^2 = 25$

c) $x^2 + y^2 - z^2 = -16$