

Dot Product with a unit vector

1. The dot product of two vectors \mathbf{x} and \mathbf{y} can be defined as

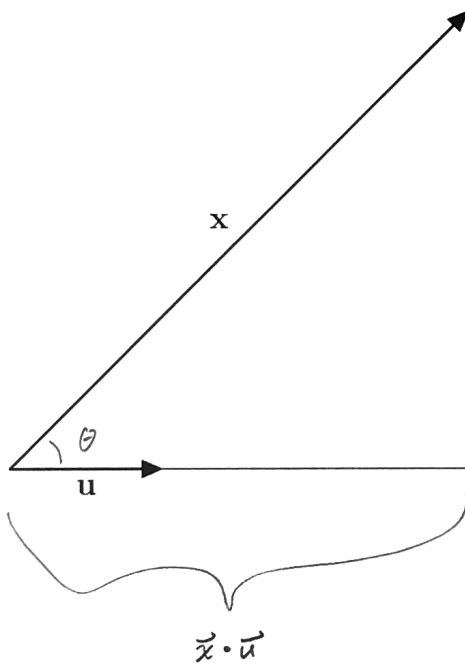
$$\mathbf{x} \cdot \mathbf{y} = |\mathbf{x}||\mathbf{y}| \cos \theta,$$

where θ is the angle between \mathbf{x} and \mathbf{y} .

- (a) Show that if $\mathbf{y} = \mathbf{u}$ is a unit vector, $\mathbf{x} \cdot \mathbf{u} = |\mathbf{x}| \cos \theta$.

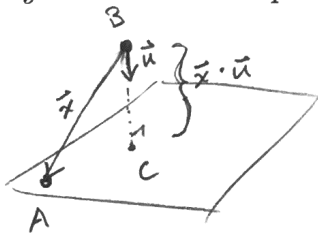
$$\begin{aligned} \vec{x} \cdot \vec{u} &= |\vec{x}| |\vec{u}| \cos \theta \\ &= |\vec{x}| \cos \theta \end{aligned}$$

- (b) Draw this length on the figure below and describe the length.



is the component of \vec{x}
in the direction of \vec{u}

2. Use the idea from the previous page to find the shortest distance from the plane $x + 2y - 2z = 6$ to the point $(-2, -2, -1)$.



choose some ^{arbitrary} point A on the plane

$$A(6, 0, 0)$$

$$\vec{x} = \vec{A} - \vec{B} = \langle 8, 2, 1 \rangle$$

$$\vec{u} = \hat{n} = \frac{\langle 1, 2, -2 \rangle}{|\langle 1, 2, -2 \rangle|} = \frac{1}{3} \langle 1, 2, -2 \rangle$$

$$d = \left| \vec{x} \cdot \vec{u} \right| = \left| \frac{1}{3} (8 + 4 - 2) \right|$$

$$= \boxed{\frac{10}{3}}$$

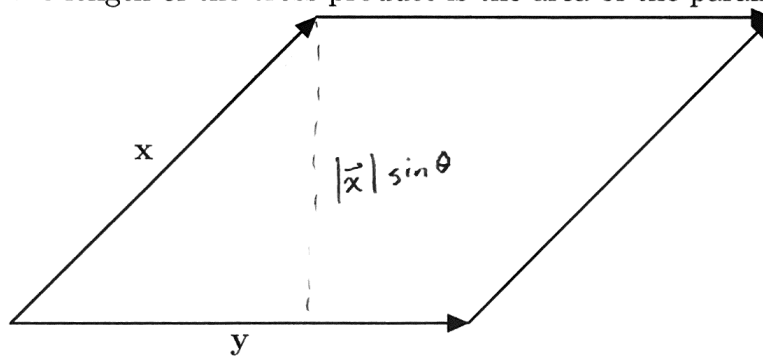
Cross Product

3. The cross product, unlike the dot product, is a vector, which has a length and a direction. The length of the cross product of two vectors \mathbf{x} and \mathbf{y} can be defined as

$$|\mathbf{x} \times \mathbf{y}| = |\mathbf{x}||\mathbf{y}| \sin \theta,$$

where θ is the angle between \mathbf{x} and \mathbf{y} . The direction of the cross product is perpendicular to both \mathbf{x} and \mathbf{y} .

- (a) Given the vectors in the figure below, label the length $|\mathbf{x}| \sin \theta$. Explain why the length of the cross product is the area of the parallelogram.



$$\begin{aligned} \text{Area} &= b \cdot h \\ &= |\vec{y}| \cdot |\vec{x}| \sin \theta \\ &= |\vec{x}| |\vec{y}| \sin \theta \\ &= |\vec{x} \times \vec{y}| \end{aligned}$$

- (b) Does the direction of $\mathbf{x} \times \mathbf{y}$ point into the page or out of the page? What about $\mathbf{y} \times \mathbf{x}$?

out

R. H. R.

4. Use your understanding of the dot product and cross product to solve the following problems.

(a) Find an equation for the line of intersection of the planes $x + 2y - 2z = 6$ and $2x - y + 2z = 12$.



↖ plane 2

$$\vec{n}_1 = \langle 1, 2, -2 \rangle$$

$$\vec{n}_2 = \langle 2, -1, 2 \rangle$$

$$\vec{v} = |\vec{n}_1 \times \vec{n}_2|$$

$$= \langle 6, -6, -5 \rangle$$

↑
plane 1

Find point of intersection:
Let $x = 0$
↓
 $y = 18, z = 15$
 $\vec{r}_0 = \langle 0, 18, 15 \rangle$

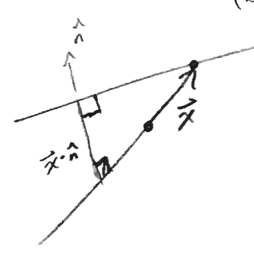
↘

$$\vec{r} = \vec{v}t + \vec{r}_0$$

$$\langle x, y, z \rangle = \langle 6, -6, -5 \rangle t + \langle 0, 18, 15 \rangle$$

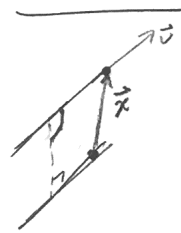
$$\begin{cases} x = 6t \\ y = -6t + 18 \\ z = -5t + 15 \end{cases}$$

(b) Find the distance between the skew lines parameterized by $\vec{r}(t) = \langle 1 + 2t, 2 - 2t, 3 + t \rangle$ and $\vec{s}(t) = \langle -1 - 2t, -2 + 2t, -3 - t \rangle$.



$$\vec{n} = \langle 2, -2, 1 \rangle \times \langle -2, 2, -1 \rangle$$

$$= \langle 0, 0, 0 \rangle \rightarrow \text{lines are actually parallel}$$



$$d = \left| \vec{x} - \frac{\vec{x} \cdot \vec{v}}{|\vec{v}|^2} \vec{v} \right|$$

$$= \left| \langle 2, 4, 6 \rangle - \frac{2}{9} \langle 2, -2, 1 \rangle \right|$$

$$= \left| \left\langle \frac{14}{9}, \frac{40}{9}, \frac{52}{9} \right\rangle \right|$$

$$= \left[\frac{5}{3} \sqrt{5} \right]$$

$$\vec{x} = \langle 1, 2, 3 \rangle - \langle -1, -2, -3 \rangle$$

$$= \langle 2, 4, 6 \rangle$$

$$\vec{v} = \langle 2, -2, 1 \rangle$$

1125
5√5
15√5

Level Sets and Graphs

5. Given the plane $x + 2y + 3z = 0$,

(a) Find a parameterization $\mathbf{r}(x, y) : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ for this plane.

$$\begin{cases} x = x \\ y = y \\ z = \frac{1}{3}(-x - 2y) \end{cases}$$

6. Write down a function $g : \mathbb{R}^3 \rightarrow \mathbb{R}^1$.

$$g(x, y, z) = xyz$$

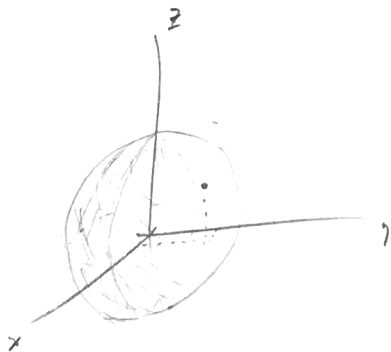
(a) The graph of g is a subset of \mathbb{R}^4 .

(b) A level set of g is a subset of \mathbb{R}^3 .

7. Find the level surface of the function $f(x, y, z) = x^2 + y^2 + z^2$ that goes through the point $(1, 2, 2)$. Draw a sketch of this surface.

$$f(1, 2, 2) = 9 \Rightarrow \boxed{x^2 + y^2 + z^2 = 9}$$

↳ sphere



8. Given the function $g(x, y, z) = x^2 + y^2 - z^2$.

- (a) Find the level surface that goes through the point $(3, 4, 5)$.
- (b) Find the level surface that goes through the point $(5, 0, 0)$.
- (c) Find the level surface that goes through the point $(0, 0, 4)$.
- (d) Draw a sketch of all three of these surfaces on the same axes.

a) $x^2 + y^2 - z^2 = 0$

b) $x^2 + y^2 - z^2 = 25$

c) $x^2 + y^2 - z^2 = -16$

