Parameterization, Curvature

1. Given the parameterization

$$\mathbf{r}(t) = \langle 2\cos(t), 3\sin(t)\rangle, \quad 0 \le t \le 2\pi.$$

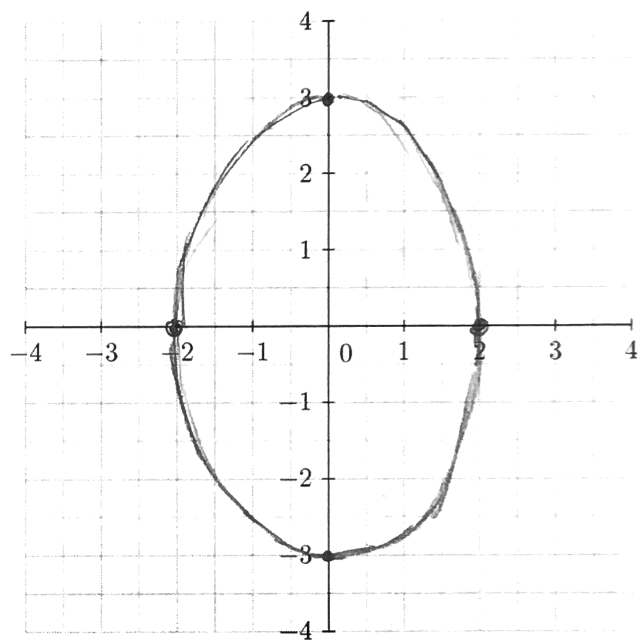
(a) Show that $x(t)$ and $y(t)$ satisfy the equation for the ellipse

$$\frac{x^2}{4} + \frac{y^2}{9} = 1.$$

$\frac{x^2}{4} + \frac{y^2}{9} = \frac{4\cos^2 t}{\cancel{4}} + \frac{9\sin^2 t}{\cancel{9}}$

$= \cos^2 t + \sin^2 t = 1$ ✓

(b) Draw this ellipse on the axes below.



(c) Find $\mathbf{v} = \mathbf{r}'(0)$ and $\mathbf{a} = \mathbf{r}''(0)$ and find $\kappa = \frac{|\mathbf{v} \times \mathbf{a}|}{|\mathbf{v}|^3}$ at $t = 0.$,

$\vec{v}(t) = \langle -2\sin t, 3\cos t\rangle \qquad \vec{v}(0) = \langle 0, 3\rangle$

$\vec{a}(t) = \langle -2\cos t, -3\sin t\rangle \qquad \vec{a}(0) = \langle -2, 0\rangle$

$$\kappa = \frac{|\langle 0,3\rangle \times \langle -2,0\rangle|}{|\langle 0,3\rangle|^3} = \frac{6}{27} = \boxed{\frac{2}{9}}$$

2. Given the parameterization for a spiral,

$$\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle.$$

(a) Find the unit tangent vector **T** and the unit normal vector **N** at time $t = 1$. [Hint: you can find the normal direction by taking $\mathbf{v} \times \mathbf{a} \times \mathbf{v}$.]

$\vec{v} = \langle -\sin t, \cos t, 1 \rangle \qquad \vec{a} = \langle -\cos t, -\sin t, 0 \rangle$

$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle -\sin t, \cos t, 1 \rangle}{\sqrt{1+1}} = \boxed{\frac{1}{\sqrt{2}} \langle -\sin t, \cos t, 1 \rangle}$

$\vec{n} = \vec{v} \times \vec{a} \times \vec{v}$

$= \langle +\sin t, -\cos t, 1 \rangle \times \langle -\sin t, \cos t, 1 \rangle$

$= \langle -2\cos t, 2\sin t, 0 \rangle \Rightarrow \vec{N} = \boxed{\langle -\cos t, \sin t, 0 \rangle}$

(b) Find an equation for a line tangent to the spiral at time t_0.

$\vec{r} = \vec{v}t + \vec{r}_0$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\sin t \\ \cos t \\ 1 \end{pmatrix} t + \begin{pmatrix} \cos t \\ \sin t \\ t \end{pmatrix}$$

(c) Given $t_0 = \pi$, find the point at which the tangent line intersects the plane $z = 0$.

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} t + \begin{pmatrix} -1 \\ 0 \\ \pi \end{pmatrix}$$

$z = 0 \Rightarrow 0 = t + \pi$

$t = -\pi$

$$\boxed{\begin{aligned} x &= -1 \\ y &= \pi \\ z &= 0 \end{aligned}}$$

3. Give examples of the following.

(a) An equation for a cylinder so that the point $(1, 2, 3)$ is on its surface.

$$x^2 + y^2 = 5$$

(b) The point P on the plane $3x+2y+z = 6$ that is closest to the point $Q(7, 7, -1)$.

$$\vec{n} = \langle 3, 2, 1 \rangle \Rightarrow \underbrace{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} t + \begin{pmatrix} 7 \\ 7 \\ -1 \end{pmatrix}}_{\text{Line } \ell}$$

Find intersection of ℓ and plane:

$$3x + 2y + z = 3(3t+7) + 2(2t+7) + 1(t-1)$$
$$= 14t + 34 = 6$$
$$t = -2 \longrightarrow (x, y, z) = \boxed{(1, 3, -3)}$$

(c) A paraboloid that opens downwards in the z direction and intersects the xy-plane in the ellipse $\underbrace{4x^2 + 9y^2 = 36}_{z=0}$.

$$\boxed{4x^2 + 9y^2 = 36 - z}$$

(d) Two parallel lines L_1 and L_2 in the parallel planes $x + 2y + 2z = 3$ and $x + 2y + 2z = 6$ so that the distance between L_1 and L_2 is 3.

Choose $L_1 := \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix}$

By inspection,

$$L_2 := \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix}$$

4. Given the points $A(1, 2, 3)$ and $B(5, 4, -2)$.

(a) Find both the parametric and symmetric equations of the straight line connecting A and B.

$$\vec{v} = B - A = \langle 4, 2, -5 \rangle$$

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ -5 \end{pmatrix} t + \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \qquad \frac{x-1}{4} = \frac{y-2}{2} = \frac{z-3}{-5}}$$

(b) Find the plane perpendicular to this line that goes through the point $(1, 2, 4)$.

$$\vec{n} = \begin{pmatrix} 4 \\ 2 \\ -5 \end{pmatrix}$$

$$4x + 2y - 5z = 4(1) + 2(2) - 5(4)$$

$$\boxed{4x + 2y - 5z = -12}$$

(c) Find a plane that this line does not intersect.

$\vec{n}$ must be $\perp$ to $\vec{v}$

$\rightarrow$ choose $\vec{n} = (1, -2, 0)$

$$x - 2y + 0z = 0$$

$$\boxed{x - 2y = 0}$$