Limits

1. Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{x^5+y^5}$.

Let $y = mx$

$$\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{x^5+y^5} \Rightarrow \lim_{x\to 0} \frac{m^3\cancel{x^5}}{(1+m^5)\cancel{x^5}}$$

$$= \frac{m^3}{1+m^5} \Rightarrow \text{different for different } m$$

$\Downarrow$

DNE

2. Evaluate $\lim_{(x,y)\to(0,0)} \frac{x^2y^3}{x^2+y^2}$. $= \lim_{r\to 0} \frac{r^{\cancel{5}\,3}\sin^3\theta\cos^2\theta}{\cancel{r^2}}$

Let $y = r\sin\theta$, $x = r\cos\theta$

$$= \lim_{r\to 0} r^3\left(\sin^3\theta\cos^2\theta\right)$$

$$= \boxed{0}$$

Vector Functions

3. For each of the following functions, first find the dimensions of the domain and the range. [For example, $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle : \mathbb{R} \to \mathbb{R}^3$.] Then find the partial derivatives of each component with respect to t, u, and/or v.

   (a) $\mathbf{r}(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle = \langle u, v, 9 - u^2 - v^2 \rangle$

   $\mathbb{R}^2 \to \mathbb{R}^3$

   $$\frac{\partial \vec{r}}{\partial u} = \langle 1, 0, -2u \rangle$$

   $$\frac{\partial \vec{r}}{\partial v} = \langle 0, 1, -2v \rangle$$

   (b) $\mathbf{r}(t, u, v) = \langle x(t, u, v), y(t, u, v) \rangle = \langle \sin(t - 2v), \sqrt{u + 3v} \rangle$

   $\mathbb{R}^3 \to \mathbb{R}^2$

   $$\frac{\partial \vec{r}}{\partial t} = \langle \cos(t-2v), 0 \rangle$$

   $$\frac{\partial \vec{r}}{\partial u} = \left\langle 0, \frac{1}{2\sqrt{u+3v}} \right\rangle$$

   $$\frac{\partial \vec{r}}{\partial v} = \left\langle -2\cos(t-2v), \frac{3}{2\sqrt{u+3v}} \right\rangle$$

4. Given the plane $x + 2y + 3z = 0$, find a parameterization $\mathbf{r}(x, y) : \mathbb{R}^2 \to \mathbb{R}^3$ for this plane.

   $$\vec{r}(x,y) = \left\langle x, y, -\frac{x+2y}{3} \right\rangle$$

   $$z = -\frac{x+2y}{3}$$

The Chain Rule

5. *Related Rates:* Gasoline is pouring into a tank in the shape of a cone of radius 3 feet and height 4 feet. When the depth of the gasoline is 2 feet, the depth is increasing at 0.2 ft/sec. How fast is the volume of the gasoline changing at that instant?

$r = 1.5$

$$V = \frac{1}{3}\pi r^2 h \qquad \frac{dr}{dt} = \frac{3}{4}\frac{dh}{dt} = 0.15 \text{ ft/s}$$

$$\frac{dV}{dt} = \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$

$$= \frac{2}{3}\pi r h \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}$$

$$= .3\pi + .15\pi$$

$$= \boxed{.45\pi} \text{ ft}^3/\text{s}$$

6. Let $z = f(x, y)$ and let $x = r\cos\theta$ and $y = r\sin\theta$.

(a) Find $\dfrac{\partial z}{\partial r}$.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$= f_x \cos\theta + f_y \sin\theta$$

(b) Find $\dfrac{\partial^2 z}{\partial r^2}$.

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right)$$

$$= \left(\frac{\partial f_x}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f_x}{\partial y}\frac{\partial y}{\partial r}\right)\cos\theta + \left(\frac{\partial f_y}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial f_y}{\partial y}\frac{\partial y}{\partial r}\right)\sin\theta$$

$$= \boxed{f_{xx}\cos^2\theta + 2 f_{xy}\sin\theta\cos\theta + f_{yy}\sin^2\theta}$$

7. Find a function $z = f(x, y)$ with $x = r\cos\theta$ and $y = r\sin\theta$ so that

$\frac{\partial x}{\partial \theta} = -r\sin\theta \qquad \frac{\partial y}{\partial \theta} = r\cos\theta$

$$\frac{\partial^2 z}{\partial r\partial\theta} = -2xr\sin\theta\cos\theta - x^2\sin\theta + 2yr\cos\theta\sin\theta + y^2\cos\theta.$$

$$\frac{\partial^2 z}{\partial r\,\partial\theta} = \frac{\partial}{\partial\theta}\left(f_x\cos\theta + f_y\sin\theta\right)$$

$$= \Big(f_{xx}(-r\sin\theta) + f_{xy}(r\cos\theta)\Big)\cos\theta + f_x(-\sin\theta) + \Big(f_{yx}(-r\sin\theta) + f_{yy}(r\cos\theta)\Big)\sin\theta + f_y(\cos\theta)$$

product rule

$$= -f_{xx}\, r\sin\theta\cos\theta + f_{xy}(r\cos^2\theta - r\sin^2\theta) + f_{yy}\, r\sin\theta\cos\theta - f_x\sin\theta + f_y\cos\theta$$

$f_x = x^2$

$f_y = y^2$

$f_{xy} = 0 \Rightarrow \boxed{f = \frac{x^3}{3} + \frac{y^3}{3}}$