

Chapter 12 Review

1. Given the points $A(1, 2, 3)$ and $B(5, 4, -2)$.

(a) Find both the parametric and symmetric equations of the straight line connecting A and B .

$$\vec{v} = B - A = \langle 4, 2, -5 \rangle$$

$$\begin{cases} x = 4t + 1 \\ y = 2t + 2 \\ z = -5t + 3 \end{cases} \quad \frac{x-1}{4} = \frac{y-2}{2} = \frac{z-3}{-5}$$

(b) Find the plane perpendicular to this line that goes through the point $(1, 2, 4)$.

$$\vec{n} = \vec{v}$$

$$4x + 2y - 5z = 4(1) + 2(2) - 5(4)$$

$$4x + 2y - 5z = -12$$

(c) Find a plane that this line does not intersect.

$$\vec{n} \perp \vec{v} \Rightarrow \text{choose } \vec{n} = \langle 1, -2, 0 \rangle \quad (\vec{n} \cdot \vec{v} = 0)$$

$$x - 2y + 0z = 0$$

↳ check line does not intersect

$$x = 4t + 1$$

$$y = 2t + 2$$

$$(4t+1) - 2(2t+2) = 0$$

$$-3 = 0$$

✓ → doesn't intersect

2. Give examples of the following, or explain why no such example exists.

(a) An equation for a cylinder so that the point $(1, 2, 3)$ is on its surface.

See week 2

$$x^2 + y^2 = 5$$

(b) A paraboloid that opens downwards in the z direction and intersects the xy -plane in the ellipse $4x^2 + 9y^2 = 36$.

See week 2

$$z = 36 - 4x^2 - 9y^2$$

(c) Two parallel lines L_1 and L_2 in the parallel planes $x + 2y + 2z = 3$ and $x + 2y + 2z = 6$ so that the distance between L_1 and L_2 is 3.

see week 2

(d) Two skew lines L_1 and L_2 that sit in parallel planes $2x + 2y + z = 5$ and $2x + 2y + z = -4$ so that the minimum distance between L_1 and L_2 is 9.

see week 2

3. Given the parameterization $\mathbf{r}(t) = \langle \cos t, \sin t, 3t \rangle$.

(a) Find the unit tangent vector at time t .

$$\vec{T} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle -\sin t, \cos t, 3 \rangle}{\sqrt{1+9}} = \frac{1}{\sqrt{10}} \langle -\sin t, \cos t, 3 \rangle$$

(b) Find an equation for the line tangent to the curve at $t = \pi$.

$$\vec{r}' = \vec{v}t' + \vec{r}_0' \quad (\text{remember } y = mx + y_0)$$

can use $\vec{v} = \vec{T}$ if preferred

$$\vec{v}' = \vec{v}(\pi) = \langle -\sin \pi, \cos \pi, 3 \rangle = \langle 0, -1, 3 \rangle$$

$$\vec{r}_0' = \vec{r}(\pi) = \langle \cos \pi, \sin \pi, 3\pi \rangle = \langle -1, 0, 3\pi \rangle$$

$$\vec{r}' = \langle 0, -1, 3 \rangle t + \langle -1, 0, 3\pi \rangle \iff \begin{cases} x = -1 \\ y = -t \\ z = 3t + 3\pi \end{cases}$$

(c) Calculate the curvature κ of the curve at $t = \pi$.

$$\kappa = \left\| \frac{d\vec{T}}{ds} \right\|$$

$$ds = \|\vec{v}\| dt$$

$$= \left\| \frac{d\left(\frac{\vec{v}}{\|\vec{v}\|}\right)}{dt} \right\|$$

$$= \frac{1}{\|\vec{v}\|} \left\| \frac{d}{dt} \frac{\vec{v}}{\|\vec{v}\|} \right\|$$

$$= \frac{1}{\sqrt{10}} \left\| \frac{d}{dt} \left(\frac{1}{\sqrt{10}} \langle -\sin t, \cos t, 3 \rangle \right) \right\|$$

$$= \frac{1}{10} \sqrt{10}$$

$$= \frac{1}{10}$$

$$\kappa = \frac{\|\vec{r}' \times \vec{r}''\|}{\|\vec{r}'\|^3}$$

$$= \frac{\|\langle 0, -1, 3 \rangle \times \langle -1, 0, 0 \rangle\|}{\sqrt{10}^3}$$

$$= \frac{\|\langle 0, 3, 1 \rangle\|}{\sqrt{10}^3} = \frac{1}{10}$$

(d) Where does this line intersect the plane $z = 0$.

$$z = 3t = 0$$

$$t = 0$$

$$\vec{r}(0) = \langle 1, 0, 0 \rangle$$

4. For the following parameterization $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$

(a) Find the velocity and acceleration vectors at time $t = 1$, $\mathbf{v}(1) = \mathbf{r}'(1)$ and $\mathbf{a}(1) = \mathbf{r}''(1)$.

$$\begin{aligned}\vec{v}(1) &= \vec{r}'(1) = \langle 1, 2t, 3t^2 \rangle \Big|_{t=1} = \langle 1, 2, 3 \rangle \\ \vec{a}(1) &= \vec{r}''(1) = \langle 0, 2, 6t \rangle \Big|_{t=1} = \langle 0, 2, 6 \rangle\end{aligned}$$

(b) Find the unit tangent vector \mathbf{T} and the unit normal vector \mathbf{N} at time $t = 1$.

[Hint: you can find the normal direction by taking $\mathbf{v} \times \mathbf{a} \times \mathbf{v}$.]

$$\begin{aligned}\hat{\mathbf{T}} &= \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 1, 2, 3 \rangle}{\sqrt{1+4+9}} = \frac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle \\ \vec{\mathbf{N}} &= \langle 1, 2, 3 \rangle \times \langle 0, 2, 6 \rangle \times \langle 1, 2, 3 \rangle = \langle 6, -6, 2 \rangle \times \langle 1, 2, 3 \rangle \\ \hat{\mathbf{N}} &= \frac{\vec{\mathbf{N}}}{\|\vec{\mathbf{N}}\|} = \frac{\langle -11, -8, 9 \rangle}{\sqrt{121+64+81}} = \langle -22, -16, 18 \rangle \\ &= \frac{1}{\sqrt{266}} \langle -11, -8, 9 \rangle\end{aligned}$$

(c) Find the curvature κ at time $t = 1$.

$$\begin{aligned}\kappa &= \left\| \frac{d\hat{\mathbf{T}}}{ds} \right\| \\ &= \frac{1}{\|\vec{v}\|} \left\| \frac{d\hat{\mathbf{T}}}{dt} \right\| \\ &= \frac{1}{\sqrt{14}} \left\| \frac{\vec{r}' \times \vec{r}''}{\|\vec{r}'\|^2} \right\| \\ &= \frac{1}{\sqrt{14}^3} \left\| \langle 6, -6, 2 \rangle \right\| \\ &= \frac{\sqrt{38}}{\sqrt{7}} \cdot \frac{1}{14}\end{aligned}$$

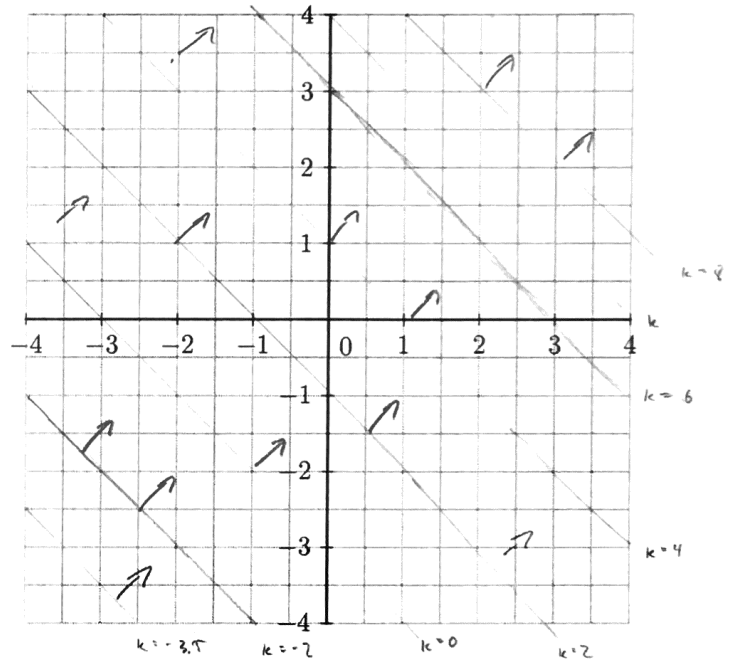
The Gradient

5. Given each of the following equations,

- (a) On the axes on the right, draw level curves of the function $f(x, y)$.
- (b) Find the gradient of f , $\langle f_x, f_y \rangle$, and draw some gradient vectors on the graph of the level curves. What is true about the gradient and level curves?

- $f(x, y) = x + y + 3 = k$

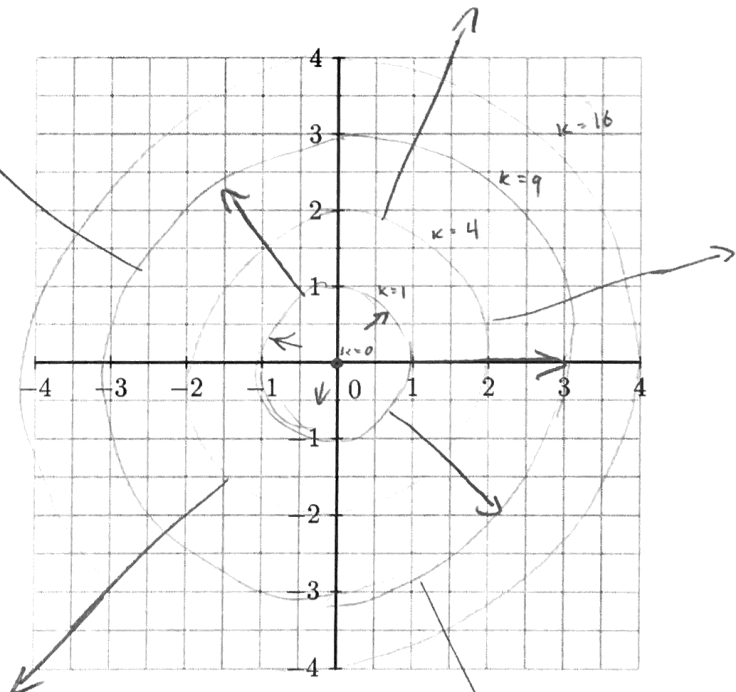
$\vec{\nabla} f = \langle 1, 1 \rangle$



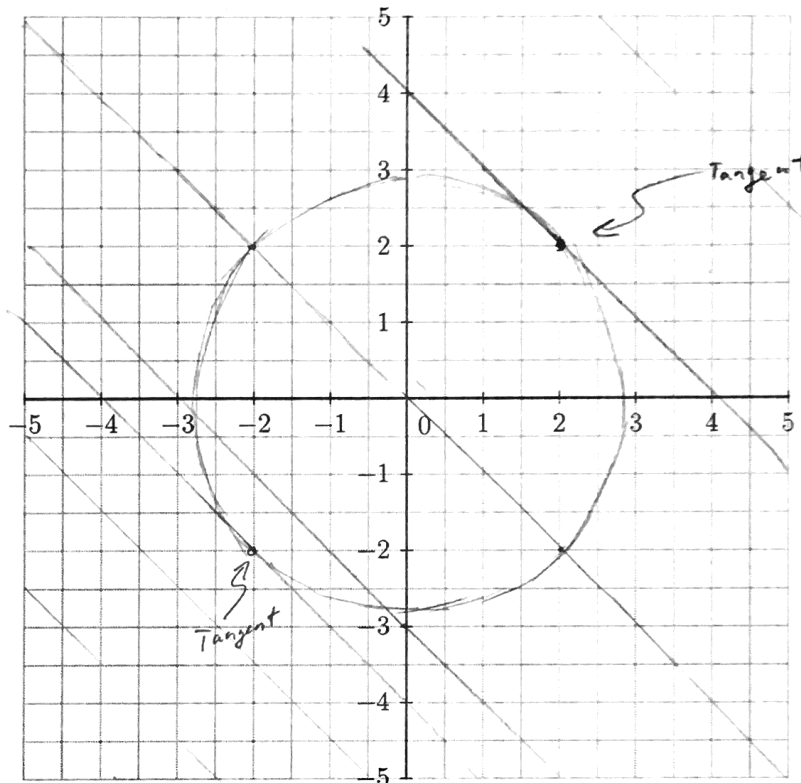
- $f(x, y) = x^2 + y^2 = k$

$\vec{\nabla} f = \langle 2x, 2y \rangle$

$\vec{\nabla} f \perp$ level sets



- (c) Show on the graph below that the points where the level curves of $f(x, y) = x + y + 3$ are tangent to the curve $x^2 + y^2 = 8$ are $(2, 2)$ and $(-2, -2)$.



- (d) Show that the gradient of $f(x, y) = x + y + 3$ and the gradient of $g(x, y) = x^2 + y^2$ are parallel (or antiparallel) at $(2, 2)$ and $(-2, -2)$.

$\vec{\nabla} f = \langle 1, 1 \rangle$	$@ (2, 2)$	$@ (-2, -2)$
$\vec{\nabla} g = \langle 2x, 2y \rangle$	$\vec{\nabla} f = \langle 1, 1 \rangle$	$\vec{\nabla} f = \langle 1, 1 \rangle$
$= 2 \langle x, y \rangle$	$\vec{\nabla} g = \langle 4, 4 \rangle$	$\vec{\nabla} g = \langle -4, -4 \rangle$
	\Downarrow	\Downarrow
	$\vec{\nabla} f = \frac{1}{4} \vec{\nabla} g$	$\vec{\nabla} f = -\frac{1}{4} \vec{\nabla} g$
	\Downarrow	\Downarrow
	$\vec{\nabla} f \parallel \vec{\nabla} g$	$\vec{\nabla} f \parallel \vec{\nabla} g$

- (e) Are these two points the maximum and minimum value of $f(x, y) = x + y + 3$ under the constraint $g(x, y) = x^2 + y^2 = 8$?

Yes.

