The Chain Rule

1. *Related Rates:* Gasoline is pouring into a tank in the shape of a cone (inverted cone? point at the bottom) of radius 3 feet and height 4 feet. When the depth of the gasoline is 2 feet, the depth is increasing at 0.2 ft/sec. How fast is the volume of the gasoline changing at that instant?

   (a) First write out the equation of the volume of a cone. Which variables are changing with time?

   $$V = \frac{1}{3}\pi r^2 h$$

   $r(t)$, $h(t)$

   (b) Differentiate implicitly with respect to time.

   $$\frac{dV}{dt} = \frac{2}{3}\pi r h \frac{dr}{dt} + \frac{1}{3}\pi r^2 \frac{dh}{dt}$$

   $$\left(= \frac{\partial V}{\partial r}\frac{dr}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} \right)$$

   (c) Solve for the rate of change of the volume.

   $h = 2$, $\frac{dh}{dt} = .2$ $\Rightarrow$ $r = 1.5$, $\frac{dr}{dt} = .15$ $\Rightarrow$ $\frac{dV}{dt} = \frac{1}{3}\pi r \left(2h\frac{dr}{dt} + r\frac{dh}{dt}\right)$

   $$= \frac{1}{3}\pi(1.5)\left(4 \cdot .15 + 1.5 \cdot .2\right)$$

   $$= \boxed{(.45)\pi}$$

2. If $z = x^2y^2$ and $x = u - v$ and $y = u + v$.

(a) Find $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$.

$$\frac{\partial x}{\partial u} = 1$$

$$\frac{\partial y}{\partial u} = 1$$

(b) Write down $\frac{\partial z}{\partial u}$ in terms of $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

(c) Solve for $\frac{\partial z}{\partial u}$ in terms of u and v.

$$\frac{\partial z}{\partial u} = (2xy^2)(1) + (2x^2y)(1)$$

$$= 2xy(x+y)$$

$$= 2(u-v)(u+v)(2u) = \boxed{4u(u^2-v^2)}$$

(d) Find $\frac{\partial^2 z}{\partial u \partial v}$.

$$\frac{\partial^2 z}{\partial u \partial v} = \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)$$

$$= \frac{\partial}{\partial v}\left(4u(u^2-v^2)\right)$$

$$= \boxed{-8uv}$$

3. Let $z = f(x, y)$ and let $x = r\cos\theta$ and $y = r\sin\theta$.

(a) Find $\dfrac{\partial z}{\partial r}$.

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

$$= \boxed{f_x \cos\theta + f_y \sin\theta}$$

(b) Find $\dfrac{\partial^2 z}{\partial r^2}$.

$$\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial r}\right)$$

$$= \frac{\partial}{\partial x}\left(f_x \cos\theta + f_y \sin\theta\right)\frac{\partial x}{\partial r} + \frac{\partial}{\partial y}\left(f_x \cos\theta + f_y \sin\theta\right)\frac{\partial y}{\partial r}$$

$$= \left(f_{xx}\cos\theta + f_{yx}\sin\theta\right)\cos\theta + \left(f_{xy}\cos\theta + f_{yy}\sin\theta\right)\sin\theta$$

$$= \boxed{f_{xx}\cos^2\theta + 2f_{xy}\sin\theta\cos\theta + f_{yy}\sin^2\theta}$$

4. Find a function $z = f(x, y)$ with $x = r\cos\theta$ and $y = r\sin\theta$ so that

$$\frac{\partial^2 z}{\partial r \partial \theta} = -2xr\sin\theta\cos\theta - x^2\sin\theta + 2yr\cos\theta\sin\theta + y^2\cos\theta.$$

See Week 5 solutions

5. Given the function

$$g(t,x,y) = y\cos(xt),$$

with $y(x) = e^x$ and $x(t) = \sin t$.

(a) Find $\dfrac{\partial g}{\partial y}$.

$$\frac{\partial g}{\partial y} = \cos(xt)$$

(b) Find $\dfrac{\partial g}{\partial x}$.

$$\frac{\partial g}{\partial x} = \frac{dy}{dx}\cos(xt) + y\frac{\partial}{\partial x}(\cos xt)$$

$$= e^x\cos(xt) + e^x(-t\sin xt)$$

$$= e^x(\cos xt - t\sin xt)$$

(c) Find $\dfrac{\partial g}{\partial t}$.

$$\frac{\partial g}{\partial t} = \frac{dy}{dt}\cos xt + y\frac{\partial}{\partial t}(\cos xt)$$

$$= \frac{dy}{dx}\frac{dx}{dt}\cos xt + y\left(\frac{\partial}{\partial t}(xt)\right)(-\sin xt)$$

$$= e^x \cos t \cos xt - y\left(\frac{dx}{dt}t + x\right)(+\sin xt)$$

$$= e^x\cos t\cos xt - y(t\cos t + x)\sin xt$$