

The Chain Rule

1. *Related Rates:* Gasoline is pouring into a tank in the shape of a cone (inverted cone? point at the bottom) of radius 3 feet and height 4 feet. When the depth of the gasoline is 2 feet, the depth is increasing at 0.2 ft/sec. How fast is the volume of the gasoline changing at that instant?

- (a) First write out the equation of the volume of a cone. Which variables are changing with time?

$$V = \frac{1}{3} \pi r^2 h$$

$$r(t), h(t)$$

- (b) Differentiate implicitly with respect to time.

$$\frac{dV}{dt} = \frac{2}{3} \pi r h \frac{dr}{dt} + \frac{1}{3} \pi r^2 \frac{dh}{dt}$$

$$\left(= \frac{\partial V}{\partial r} \frac{dr}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt} \right)$$

- (c) Solve for the rate of change of the volume.

$$\begin{aligned} h = 2 & \Rightarrow r = 1.5 \\ \frac{dh}{dt} = .2 & \Rightarrow \frac{dr}{dt} = .15 \end{aligned} \Rightarrow \frac{dV}{dt} = \frac{1}{3} \pi r \left(2h \frac{dr}{dt} + r \frac{dh}{dt} \right)$$

$$= \frac{1}{3} \pi (1.5) (4 \cdot .15 + 1.5 \cdot .2)$$

$$= \boxed{(.45) \pi}$$

2. If $z = x^2y^2$ and $x = u - v$ and $y = u + v$.

(a) Find $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$.

$$\frac{\partial x}{\partial u} = 1$$

$$\frac{\partial y}{\partial u} = 1$$

(b) Write down $\frac{\partial z}{\partial u}$ in terms of $\frac{\partial x}{\partial u}$ and $\frac{\partial y}{\partial u}$.

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}$$

(c) Solve for $\frac{\partial z}{\partial u}$ in terms of u and v .

$$\begin{aligned} \frac{\partial z}{\partial u} &= (2xy^2)(1) + (2x^2y)(1) \\ &= \cancel{2xy^2} + \cancel{2x^2y} \\ &= 2(u-v)(u+v)(2u) = \boxed{4u(u^2 - v^2)} \end{aligned}$$

(d) Find $\frac{\partial^2 z}{\partial u \partial v}$.

$$\begin{aligned} \frac{\partial^2 z}{\partial u \partial v} &= \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} \right) \\ &= \frac{\partial}{\partial v} \left(4u(u^2 - v^2) \right) \\ &= \boxed{-8uv} \end{aligned}$$

3. Let $z = f(x, y)$ and let $x = r \cos \theta$ and $y = r \sin \theta$.

(a) Find $\frac{\partial z}{\partial r}$.

$$\begin{aligned} \frac{\partial z}{\partial r} &= \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r} \\ &= \boxed{f_x \cos \theta + f_y \sin \theta} \end{aligned}$$

(b) Find $\frac{\partial^2 z}{\partial r^2}$.

$$\begin{aligned} \frac{\partial^2 z}{\partial r^2} &= \frac{\partial}{\partial r} \left(\frac{\partial z}{\partial r} \right) \\ &= \frac{\partial}{\partial x} \left(f_x \cos \theta + f_y \sin \theta \right) \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \left(f_x \cos \theta + f_y \sin \theta \right) \frac{\partial y}{\partial r} \\ &= \left(f_{xx} \cos \theta + f_{yx} \sin \theta \right) \cos \theta + \left(f_{xy} \cos \theta + f_{yy} \sin \theta \right) \sin \theta \\ &= \boxed{f_{xx} \cos^2 \theta + 2f_{xy} \sin \theta \cos \theta + f_{yy} \sin^2 \theta} \end{aligned}$$

4. Find a function $z = f(x, y)$ with $x = r \cos \theta$ and $y = r \sin \theta$ so that

$$\frac{\partial^2 z}{\partial r \partial \theta} = -2xr \sin \theta \cos \theta - x^2 \sin \theta + 2yr \cos \theta \sin \theta + y^2 \cos \theta.$$

See week 5 solutions

5. Given the function

$$g(t, x, y) = y \cos(xt),$$

with $y(x) = e^x$ and $x(t) = \sin t$.

(a) Find $\frac{\partial g}{\partial y}$.

$$\frac{\partial g}{\partial y} = \cos(xt)$$

(b) Find $\frac{\partial g}{\partial x}$.

$$\begin{aligned} \frac{\partial g}{\partial x} &= \frac{\partial y}{\partial x} \cos(xt) + y \frac{\partial}{\partial x} (\cos xt) \\ &= e^x \cos(xt) + e^x (-t \sin xt) \\ &= e^x (\cos xt - t \sin xt) \end{aligned}$$

(c) Find $\frac{\partial g}{\partial t}$.

$$\begin{aligned} \frac{\partial g}{\partial t} &= \frac{\partial y}{\partial t} \cos xt + y \frac{\partial}{\partial t} (\cos xt) \\ &= \cancel{\frac{\partial y}{\partial t} \cos xt} + y \left(\frac{\partial}{\partial t} (xt) \right) (-\sin xt) \\ &= e^x \cancel{\cos t} \cos xt - y \left(\frac{dx}{dt} t + x \right) (-\sin xt) \\ &= e^x \cos t \cos xt - y (t \cos t + x) \sin xt \end{aligned}$$