The Chain Rule

1. Related Rates: Gasoline is pouring into a tank in the shape of a cone (inverted cone? point at the bottom) of radius 3 feet and height 4 feet. When the depth of the gasoline is 2 feet, the depth is increasing at 0.2 ft/sec. How fast is the volume of the gasoline changing at that instant?

   (a) First write out the equation of the volume of a cone. Which variables are changing with time?

   \[ V = \frac{1}{3} \pi r^2 h \]

   \[ r(t), \ h(t) \]

   (b) Differentiate implicitly with respect to time.

   \[ \frac{dV}{dt} = \frac{2}{3} \pi r \frac{dr}{dt} h + \frac{1}{3} \pi r^2 \frac{dh}{dt} \]

   \[ \left( \frac{\partial V}{\partial r} \frac{dr}{dt} + \frac{\partial V}{\partial h} \frac{dh}{dt} \right) \]

   (c) Solve for the rate of change of the volume.

   \[ \frac{dh}{dt} = 0.2 \Rightarrow r = 1.5 \]

   \[ \frac{dl}{dt} = 0.15 \Rightarrow \frac{dV}{dt} = \frac{1}{3} \pi r \left( 2.5 \frac{dr}{dt} + r \frac{dh}{dt} \right) \]

   \[ = \frac{1}{3} \pi (1.5) (4 \cdot 0.5 + 1.5 \cdot 0.3) \]

   \[ = \left( \frac{45}{10} \right) \pi \]
2. If \( z = x^2 y^2 \) and \( x = u - v \) and \( y = u + v \).

(a) Find \( \frac{\partial x}{\partial u} \) and \( \frac{\partial y}{\partial u} \).

\[
\frac{\partial x}{\partial u} = 1
\]

\[
\frac{\partial y}{\partial u} = 1
\]

(b) Write down \( \frac{\partial z}{\partial u} \) in terms of \( \frac{\partial z}{\partial x} \) and \( \frac{\partial z}{\partial y} \).

\[
\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial u} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial u}
\]

(c) Solve for \( \frac{\partial z}{\partial u} \) in terms of \( u \) and \( v \).

\[
\frac{\partial z}{\partial u} = \left( 2xy^2 \right) \left( 1 \right) + \left( 2x^2y \right) \left( 1 \right)
\]

\[
= 2xy (x + y)
\]

\[
= 2 (u - v)(u + v)(2u) = 4u (u^2 - v^2)
\]

(d) Find \( \frac{\partial^2 z}{\partial u \partial v} \).

\[
\frac{\partial^2 z}{\partial u \partial v} = \frac{\partial}{\partial v} \left( \frac{\partial z}{\partial u} \right)
\]

\[
= \frac{\partial}{\partial v} \left( 4u (u^2 - v^2) \right)
\]

\[
= -8uv
\]
3. Let \( z = f(x, y) \) and let \( x = r \cos \theta \) and \( y = r \sin \theta \).

(a) Find \( \frac{\partial z}{\partial r} \).

\[
\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \frac{\partial y}{\partial r}
\]

\[= f_x \cos \theta + f_y \sin \theta \]

(b) Find \( \frac{\partial^2 z}{\partial r^2} \).

\[
\frac{\partial^2 z}{\partial r^2} = \frac{\partial}{\partial r} \left( \frac{\partial z}{\partial r} \right)
\]

\[= \frac{\partial}{\partial x} \left( f_x \cos \theta + f_y \sin \theta \right) \frac{\partial x}{\partial r} + \frac{\partial}{\partial y} \left( f_x \cos \theta + f_y \sin \theta \right) \frac{\partial y}{\partial r}
\]

\[= (f_{xx} \cos \theta + f_{yx} \sin \theta) \cos \theta + (f_{xy} \cos \theta + f_{yy} \sin \theta) \sin \theta
\]

\[= f_{xx} \cos^2 \theta + 2 f_{xy} \sin \theta \cos \theta + f_{yy} \sin^2 \theta
\]

4. Find a function \( z = f(x, y) \) with \( x = r \cos \theta \) and \( y = r \sin \theta \) so that

\[
\frac{\partial^2 z}{\partial r \partial \theta} = -2xr \sin \theta \cos \theta - x^2 \sin \theta + 2yr \cos \theta \sin \theta + y^2 \cos \theta.
\]

See Week 5 solutions.
5. Given the function

\[ g(t, x, y) = y \cos(xt), \]

with \( y(x) = e^x \) and \( x(t) = \sin t \).

(a) Find \( \frac{\partial g}{\partial y} \).

\[ \frac{\partial g}{\partial y} = \cos(\pi t) \]

(b) Find \( \frac{\partial g}{\partial x} \).

\[ \frac{\partial g}{\partial x} = \frac{dy}{dx} \cos(xt) + y \frac{d}{dx}(\cos xt) \]

\[ = e^x \cos(\pi t) + e^x (-\sin xt) \]

\[ = e^x \left( \cos xt - \sin xt \right) \]

(c) Find \( \frac{\partial g}{\partial t} \).

\[ \frac{\partial g}{\partial t} = \frac{dy}{dt} \cos xt + y \frac{d}{dt}(\cos xt) \]

\[ = \frac{dy}{dx} \frac{dx}{dt} \cos xt + y \left( \frac{\partial}{\partial t}(\cos xt) \right)(-\sin xt) \]

\[ = e^x \cos xt \cos xt - y \left( \frac{dx}{dt} t + \pi \right) (\pi \sin xt) \]

\[ = e^x \cos xt \cos xt - y (t \cos xt + \pi) \sin xt \]