

Optimization

1. Given the function,

$$f(x, y) = 2xy - x^2y - y^2x.$$

(a) Find all the critical points. The graph below might be helpful.

$$\nabla f = \langle 2y - 2xy - y^2, \\ 2x - x^2 - 2xy \rangle$$

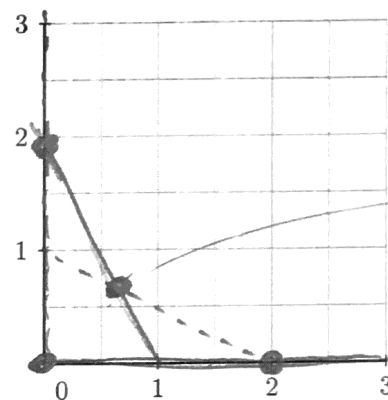
$$2y - 2xy - y^2 = 0$$

$$y(y + 2x - 2) = 0$$

$$y = 0, 2 - 2x$$

$$2x - x^2 - 2xy = 0$$

$$x = 0, 2 - 2y$$



$$x = y \\ x = 2 - 2x \\ x = \frac{2}{3}$$

$$(0, 0)$$

$$(0, 2)$$

$$(2, 0)$$

$$\left(\frac{2}{3}, \frac{2}{3}\right)$$

$$f(x, y) = 2xy - x^2y - y^2x$$

(b) Classify each critical point using the Hessian matrix.

$$\frac{\partial^2 f}{\partial x^2} = -2y \quad \frac{\partial^2 f}{\partial y^2} = -2x \quad \frac{\partial^2 f}{\partial x \partial y} = 2 - 2x - 2y$$

$$H = \begin{vmatrix} -2y & 2-2x-2y \\ 2-2x-2y & -2x \end{vmatrix} = 4xy - 4(x+y-1)^2$$

$$H(0, 0) = -4 \rightarrow \text{saddle}$$

$$H(0, 2) = -4 \rightarrow \text{saddle}$$

$$H(2, 0) = -4 \rightarrow \text{saddle}$$

$$H\left(\frac{2}{3}, \frac{2}{3}\right) = \frac{4}{3} \rightarrow \text{local max}$$

$$f_{xx}\left(\frac{2}{3}, \frac{2}{3}\right) = -\frac{4}{3}$$

$$f(x, y) = 2xy - x^2y - y^2x$$

- (c) Find the maximum and minimum values of the function in the region within the triangle shown below, including the edges of the triangle.

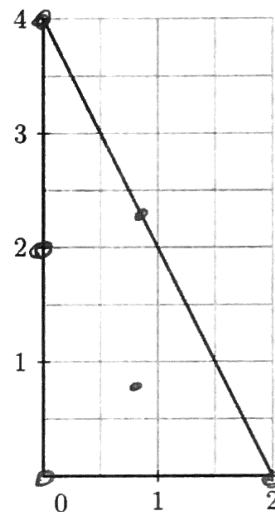
Find extrema along $y = 4 - 2x$:

$$f(x, 4-2x) = 2x(4-2x) - x^2(4-2x) - (4-2x)^2x$$

$$f'(x, 4-2x) = 8 - 8x - 8x + 6x^2 - 12x^2 + 32x - 16$$

$$= -8 + 16x - 6x^2 = 0$$

$$x = \frac{8 \pm \sqrt{64 - 48}}{6} = 2, \frac{2}{3}$$



$$f(0, 0) = 0$$

$$f(0, 2) = 0$$

$$f(2, 0) = 0$$

$$f(4, 0) = 0$$

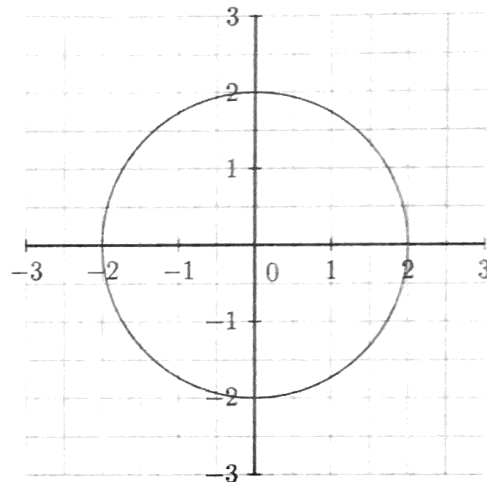
$$f\left(\frac{2}{3}, \frac{2}{3}\right) = \frac{8}{9} - \frac{16}{27} = \frac{8}{27}$$

$$\text{Max @ } f\left(\frac{2}{3}, \frac{2}{3}\right) = \frac{8}{27}$$

$$f\left(\frac{2}{3}, \frac{8}{3}\right) = -\frac{64}{27}$$

$$\text{Min @ } f\left(\frac{2}{3}, \frac{8}{3}\right) = -\frac{64}{27}$$

2. Given the function $f(x, y) = 5x^2 - 6xy + 5y^2$ with constraint $x^2 + y^2 = 4$, shown below.



- (a) Find all points where the gradient of f is parallel (or anti-parallel) to the gradient of $g(x, y) = x^2 + y^2$.

$$\vec{\nabla} f = \langle 10x - 6y, 10y - 6x \rangle$$

$$\vec{\nabla} g = \langle 2x, 2y \rangle$$

$$\vec{\nabla} f = \lambda \vec{\nabla} g$$

$$10x - 6y = \lambda 2x$$

$$(5 - \lambda)x = 6y$$

$$10y - 6x = \lambda 2y$$

$$(5 - \lambda)y = 6x$$

$$\boxed{\pm x = y}$$

- (b) Find all points on the constraint $x^2 + y^2 = 4$ where the gradient of f is parallel (or anti-parallel) to the gradient of $g(x, y) = x^2 + y^2$.

$$\begin{cases} x = \pm y \\ x^2 + y^2 = 4 \end{cases} \Rightarrow \begin{cases} x = \pm\sqrt{2} \\ y = \pm\sqrt{2} \end{cases}$$

- (c) Find the maximum and minimum values of $f(x, y)$ under the constraint $x^2 + y^2 = 4$.

$$f(-\sqrt{2}, -\sqrt{2}) = 8$$

$$f(-\sqrt{2}, \sqrt{2}) = 32$$

$$f(\sqrt{2}, -\sqrt{2}) = 32$$

$$f(\sqrt{2}, \sqrt{2}) = 8$$

$$\begin{array}{l} \text{Max value: } 32 \\ \text{min value: } 8 \end{array}$$