

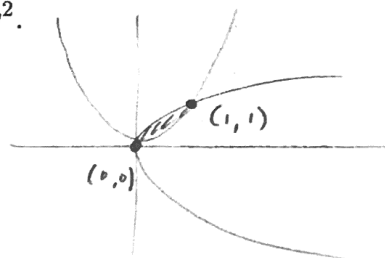
## Double Integrals

1. Find the area between the curves  $y = x^2$  and  $x = y^2$ .

$$A = \int_0^1 \int_{x^2}^{\sqrt{x}} 1 \, dy \, dx$$

$$= \int_0^1 (\sqrt{x} - x^2) \, dx$$

$$= \frac{2}{3} - \frac{1}{3} = \boxed{\frac{1}{3}}$$



2. Find the area between the curves  $x + 2y = 1$  and  $x = y^2 - 2$ .

Find intersection points:

$$1 - 2y = y^2 - 2$$

$$0 = y^2 + 2y - 3$$

$$y = 1, -3$$

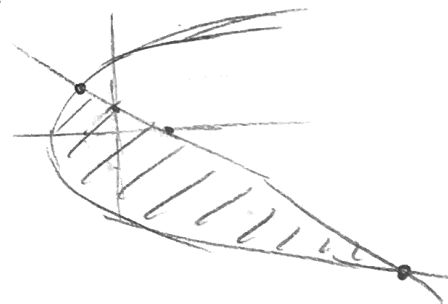
$$A = \int_{-3}^1 \int_{y^2-2}^{1-2y} 1 \, dx \, dy$$

$$= \int_{-3}^1 (1 - 2y - y^2 + 2) \, dy$$

$$= \left[ -\frac{1}{3}y^3 - y^2 + 3y \right]_{-3}^1$$

$$= -\frac{1}{3} - 1 + 3 - 9 + 9 + 9$$

$$= \boxed{\frac{10}{3}} = \frac{32}{3}$$



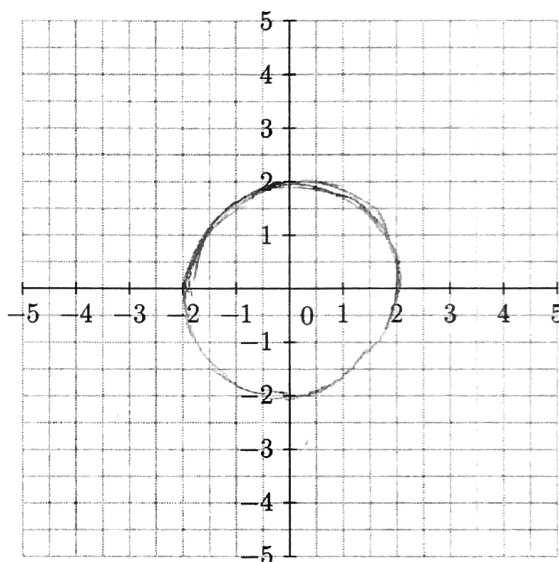
3. Given the equations below,

- Draw a picture of the curve formed by the equation.
- Write down an integral in rectangular coordinates that would give the area inside the curve.
- Convert the equation to polar coordinates.
- Write down an integral in polar coordinates that would give the area inside the curve.

(a)  $x^2 + y^2 = 4$

$$A = \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 1 \, dy \, dx$$

$$= \int_0^2 \int_0^{2\pi} r \, d\theta \, dr$$



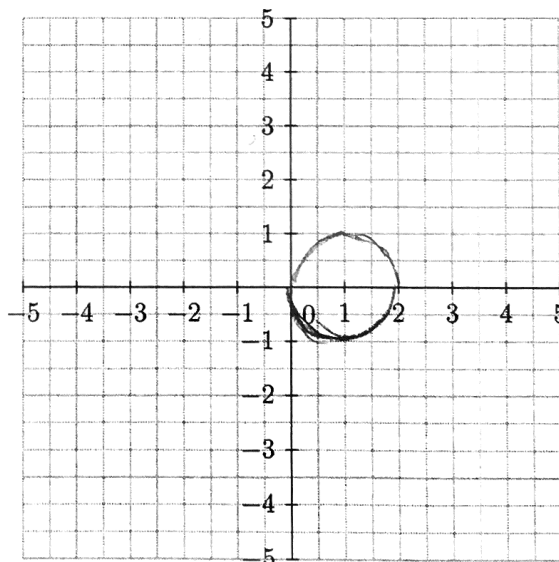
(b)  $(x-1)^2 + y^2 = 1$

$$A = \int_0^2 \int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} 1 \, dy \, dx$$

$$r = 2 \cos \theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{2 \cos \theta} r \, dr \, d\theta$$

$$\begin{aligned} \cos 2\theta &= 2\cos^2\theta - 1 \\ &= 1 - 2\sin^2\theta \end{aligned}$$



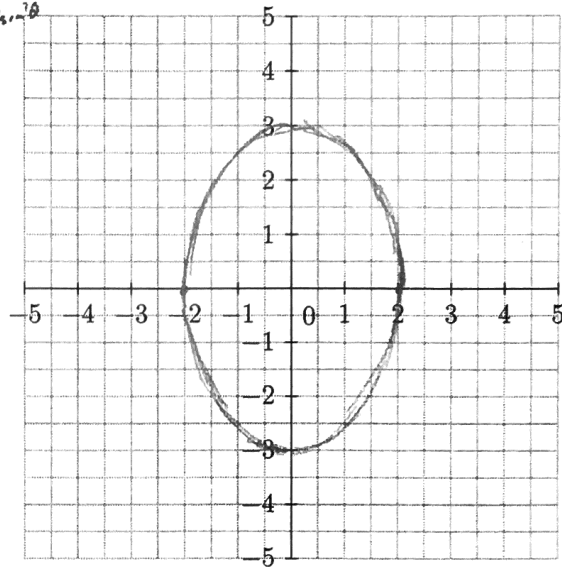
$$(c) \quad 9x^2 + 4y^2 = 36$$

$$A = \int_{-2}^2 \int_{-\sqrt{9-\frac{9}{4}x^2}}^{\sqrt{9-\frac{9}{4}x^2}} 1 \, dy \, dx$$

$$= \int_0^{2\pi} \int_0^{\frac{6}{\sqrt{9\cos^2\theta + 4\sin^2\theta}}} r \, dr \, d\theta$$

$$9r^2 \cos^2\theta + 4r^2 \sin^2\theta = 36$$

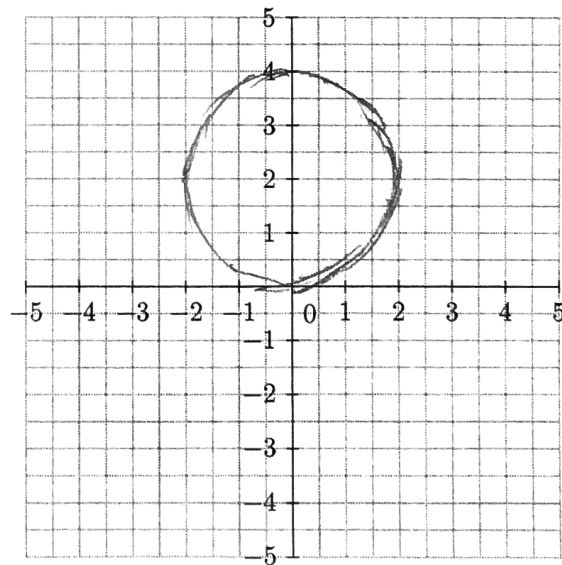
$$r = \frac{6}{\sqrt{9\cos^2\theta + 4\sin^2\theta}}$$



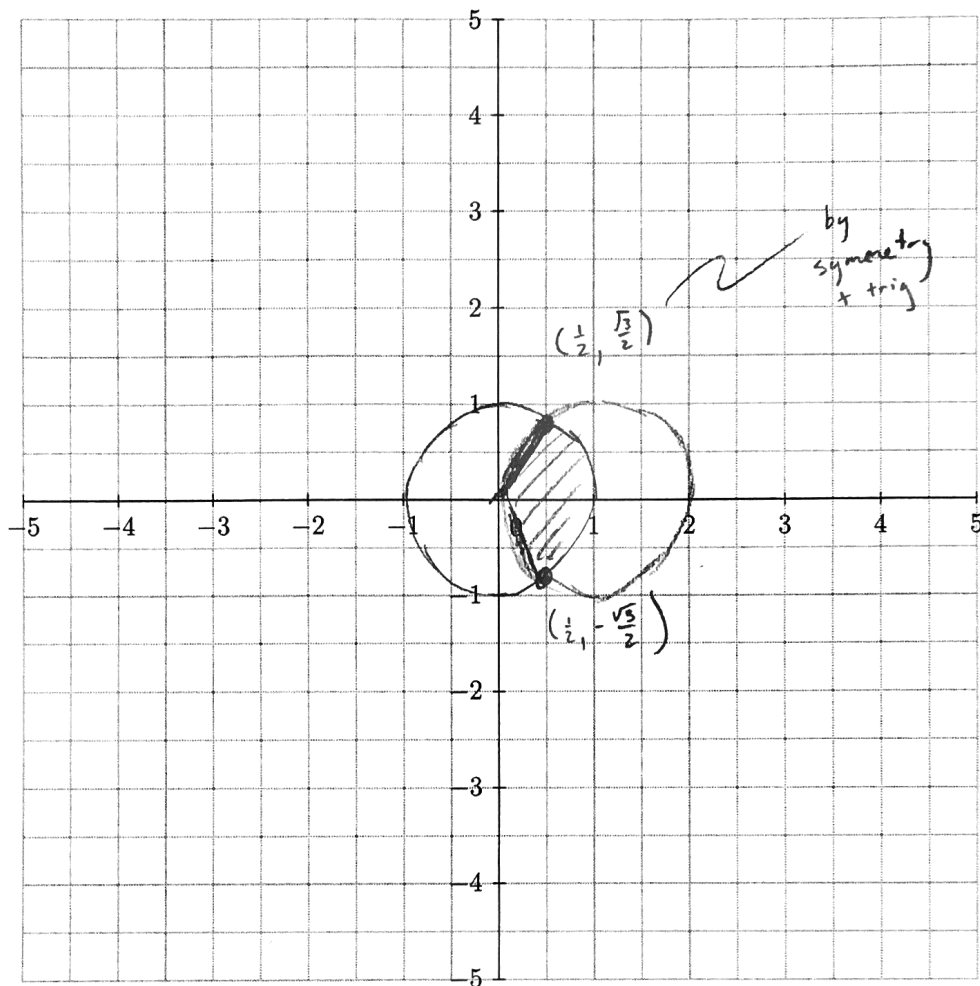
$$(d) \quad x^2 + (y-2)^2 = 4$$

$$A = \int_{-2}^2 \int_{2-\sqrt{4-x^2}}^{2+\sqrt{4-x^2}} 1 \, dy \, dx$$

$$= \int_0^{\pi} \int_0^{4\sin\theta} r \, dr \, d\theta$$



4. Draw a picture of the graphs of the following  $r = 2 \cos \theta$  and  $r = 1$  on the axes below. Write down the points  $(x, y)$  where the graphs intersect.



Find the area inside both curves.

$$\begin{aligned}
 A &= 2 \int_{\pi/3}^{\pi/2} \int_0^{2 \cos \theta} r \, dr \, d\theta + \int_0^{\pi/3} \int_0^1 r \, dr \, d\theta \\
 &= \int_0^{\pi/3} 1 \, d\theta + \int_{\pi/3}^{\pi/2} 4 \cos^2 \theta \, d\theta \\
 &= \frac{\pi}{3} + 4 \left[ \frac{1 + \frac{1}{2} \sin 2\theta}{2} \right]_{\pi/3}^{\pi/2} \\
 &= \frac{\pi}{3} + 2 \left( \frac{\pi}{2} - \frac{\pi}{3} + 0 - \frac{\sqrt{3}}{4} \right) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}
 \end{aligned}$$