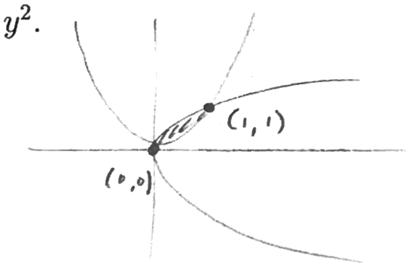
Double Integrals

1. Find the area between the curves $y = x^2$ and $x = y^2$.

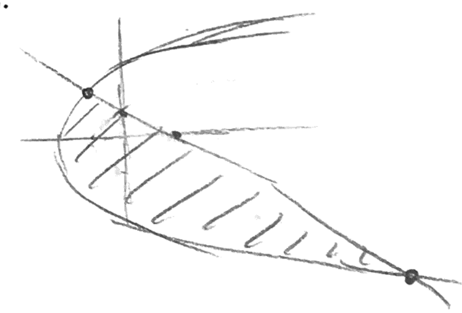


$$A = \int_0^1 \int_{x^2}^{\sqrt{x}} 1 \, dy \, dx$$

$$= \int_0^1 (\sqrt{x} - x^2) \, dx$$

$$= \frac{2}{3} - \frac{1}{3} = \boxed{\frac{1}{3}}$$

2. Find the area between the curves $x + 2y = 1$ and $x = y^2 - 2$.



Find intersection points:

$$1 - 2y = y^2 - 2$$

$$0 = y^2 + 2y - 3$$

$$y = 1, -3$$

$$A = \int_{-3}^{1} \int_{y^2-2}^{1-2y} 1 \, dx \, dy$$

$$= \int_{-3}^{1} (1 - 2y - y^2 + 2) \, dy$$

$$= \left[-\frac{1}{3} y^3 - y^2 + 3y \right]_{-3}^{1}$$

$$= -\frac{1}{3} - 1 + 3 \cancel{-9} \cancel{+9} + 9$$

$$= \boxed{10\frac{2}{3}} = \frac{32}{3}$$

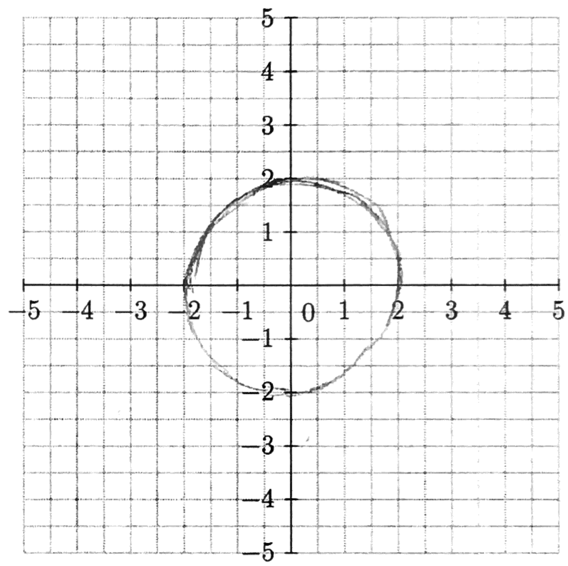
3. Given the equations below,

- Draw a picture of the curve formed by the equation.
- Write down an integral in rectangular coordinates that would give the area inside the curve.
- Convert the equation to polar coordinates.
- Write down an integral in polar coordinates that would give the area inside the curve.

(a) $x^2 + y^2 = 4$

$$A = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 1 \, dy \, dx$$

$$= \int_{0}^{2} \int_{0}^{2\pi} r \, d\theta \, dr$$



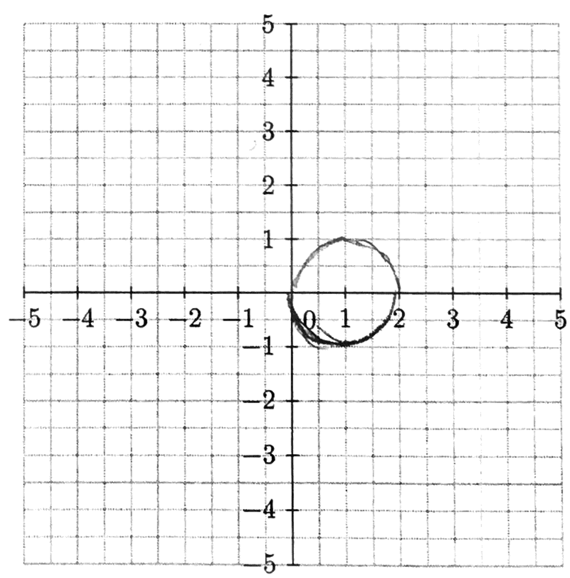
$\cos 2\theta = 2\cos^2\theta - 1$
$= 1 - 2\sin^2\theta$

(b) $(x-1)^2 + y^2 = 1$

$$A = \int_{0}^{2} \int_{-\sqrt{1-(x-1)^2}}^{\sqrt{1-(x-1)^2}} 1 \, dy \, dx$$

$r = 2\cos\theta$

$$= \int_{-\frac{\pi}{2}}^{\pi/2} \int_{0}^{2\cos\theta} 1 \cdot r \, dr \, d\theta$$



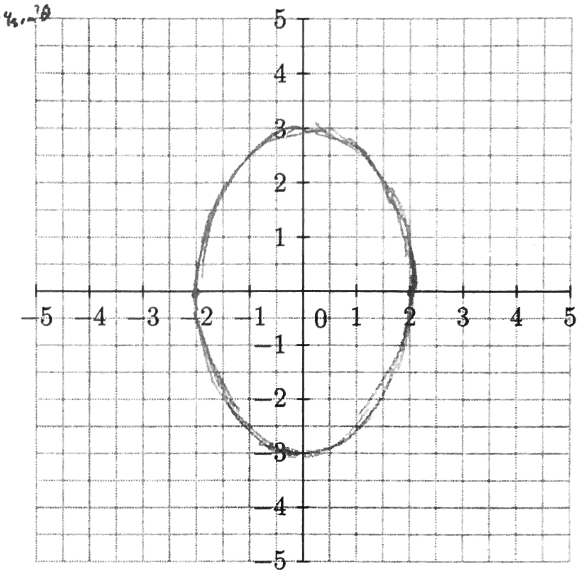
(c) $9x^2 + 4y^2 = 36$

$9r^2\cos^2\theta + 4r^2\sin^2\theta = 36$

$r = \frac{6}{\pm\sqrt{9\cos^2\theta + 4\sin^2\theta}}$

$$A = \int_{-2}^{2} \int_{-\sqrt{9-\frac{9}{4}x^2}}^{\sqrt{9-\frac{9}{4}x^2}} 1 \, dy\, dx$$

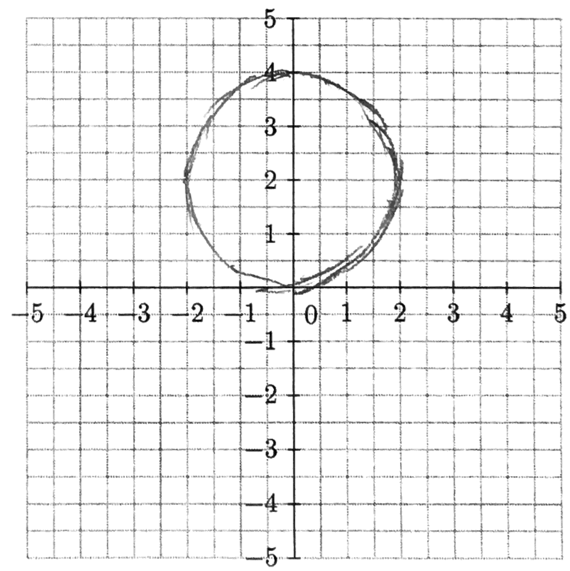
$$= \int_{0}^{2\pi} \int_{0}^{\frac{6}{\sqrt{9\cos^2\theta + 4\sin^2\theta}}} r \, dr\, d\theta$$



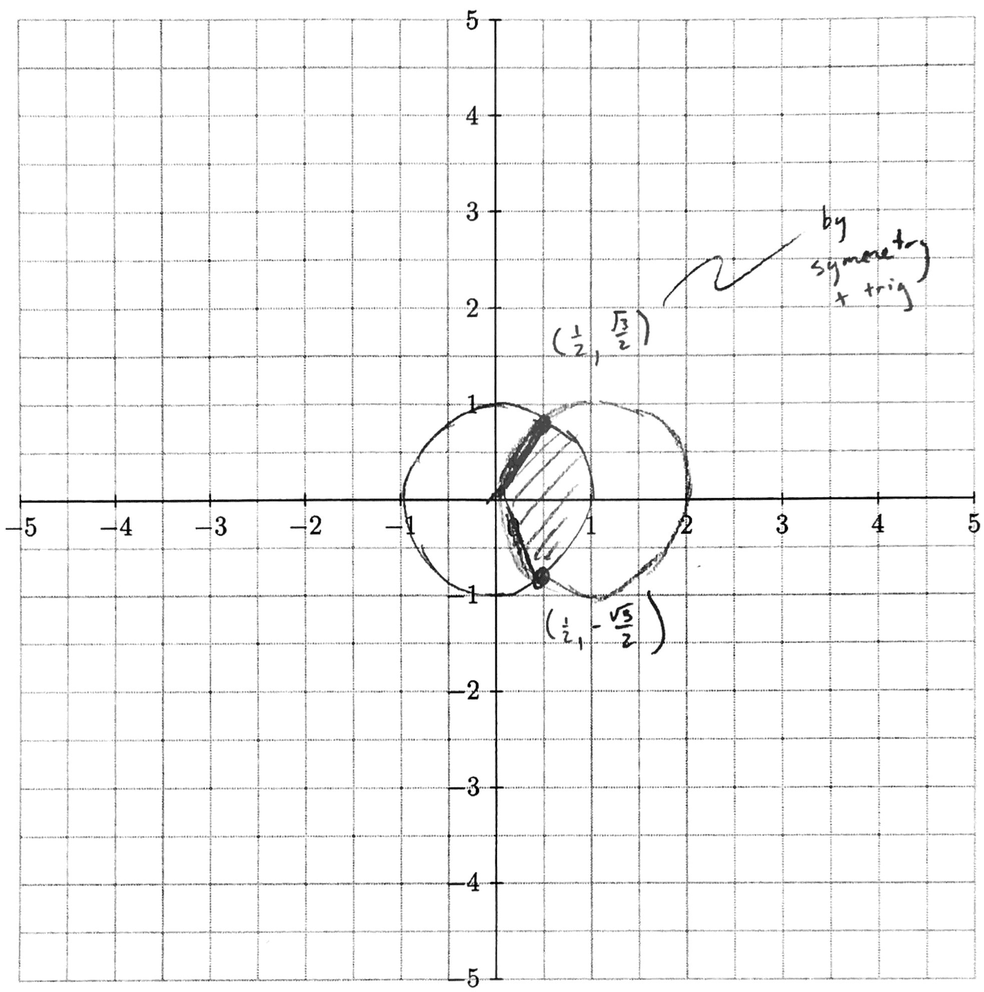
(d) $x^2 + (y-2)^2 = 4$

$$A = \int_{-2}^{2} \int_{2-\sqrt{4-x^2}}^{2+\sqrt{4-x^2}} 1 \, dy\, dx$$

$$= \int_{0}^{\pi} \int_{0}^{4\sin\theta} r \, dr\, d\theta$$



4. Draw a picture of the graphs of the following $r = 2\cos\theta$ and $r = 1$ on the axes below. Write down the points (x, y) where the graphs intersect.



Find the area inside both curves.

$$A = 2\left(\int_{\pi/3}^{\pi/2} \int_0^{2\cos\theta} r \, dr\, d\theta + \int_0^{\pi/3}\int_0^1 r\, dr\, d\theta\right)$$

$$= \int_0^{\pi/3} 1 \, d\theta + \int_{\pi/3}^{\pi/2} 4\cos^2\theta \, d\theta$$

$$= \frac{\pi}{3} + 4\left[\frac{1 + \frac{1}{2}\sin 2\theta}{2}\right]_{\pi/3}^{\pi/2}$$

$$= \frac{\pi}{3} + 2\left(\frac{\pi}{2} - \frac{\pi}{3} + 0 - \frac{\sqrt{3}}{4}\right) = \boxed{\frac{2\pi}{3} - \frac{\sqrt{3}}{2}}$$