Double Integrals

1. Find the area between the curves \( y = x^2 \) and \( x = y^2 \).

\[
A = \int_{0}^{1} \int_{y^2}^{\sqrt{x}} 1 \, dy \, dx
= \int_{0}^{1} \left( \sqrt{x} - y^2 \right) \, dx
= \frac{2}{3} - \frac{1}{3} = \frac{1}{3}
\]

2. Find the area between the curves \( x + 2y = 1 \) and \( x = y^2 - 2 \).

Find intersection points:

\[
\begin{align*}
1 - 2y &= y^2 - 2 \\
0 &= y^2 + 2y - 3 \\
y &= 1, -3
\end{align*}
\]

\[
A = \int_{-3}^{1} \int_{y-2}^{1-2y} 1 \, dx \, dy
= \int_{-3}^{1} \left( 1 - 2y - y^2 + 2 \right) \, dy
= \left[ \frac{1}{3}y^3 - y^2 + 3y \right]_{-3}
= -\frac{1}{3} - 1 + 3 - 9 + 9
= \frac{10}{3}
\]
3. Given the equations below,

- Draw a picture of the curve formed by the equation.
- Write down an integral in rectangular coordinates that would give the area inside the curve.
- Convert the equation to polar coordinates.
- Write down an integral in polar coordinates that would give the area inside the curve.

\[(a) \quad x^2 + y^2 = 4\]

\[
A = \int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} 1 \, dy \, dx
\]

\[
= \int_{0}^{2\pi} \int_{0}^{2} r \, dr \, d\theta
\]

\[(b) \quad (x - 1)^2 + y^2 = 1\]

\[
A = \int_{0}^{2\pi} \int_{0}^{1/(1-\cos^2\theta)} 1 \, dy \, d\alpha
\]

\[
= \int_{0}^{\pi/2} \int_{0}^{2\cos\theta} r \, dr \, d\theta
\]
(c) \[9x^2 + 4y^2 = 36\]

\[
A = \int \int_{\frac{\sqrt{9-x^2}}{2}}^{2} 1 \, dy \, dx
\]

\[
= \int_{2}^{\frac{\sqrt{9-x^2}}{2}} \int_{\phi}^{2\pi} r \, dr \, d\theta
\]

\[
r = \frac{6}{\sqrt{9-r^2}}
\]

(d) \[x^2 + (y-2)^2 = 4\]

\[
A = \int \int_{\frac{\sqrt{4-y^2}}{2}}^{2} 1 \, dy \, dx
\]

\[
= \int_{2}^{\frac{\sqrt{4-y^2}}{2}} \int_{\phi}^{2\pi} r \, dr \, d\theta
\]

\[
r = y \sin \theta
\]
4. Draw a picture of the graphs of the following $r = 2 \cos \theta$ and $r = 1$ on the axes below. Write down the points $(x, y)$ where the graphs intersect.

Find the area inside both curves.

$$A = \int_{\pi/3}^{\pi/2} \int_0^r r \, dr \, d\theta + \int_{\pi/3}^{\pi/2} \int_0^1 r \, dr \, d\theta$$

$$= \int_{\pi/3}^{\pi/2} \left[ \frac{r^2}{2} \right]_0^r \, d\theta + \int_{\pi/3}^{\pi/2} \left[ \frac{\theta}{2} \cos^2 \theta \right] \, d\theta$$

$$= \frac{\pi}{3} + \frac{1}{4} \left[ 1 + \frac{1}{2} \sin 2\theta \right]_{\pi/3}^{\pi/2}$$

$$= \frac{\pi}{3} + 2 \left( \frac{\pi}{3} - \frac{\pi}{2} + 0 - \frac{\sqrt{3}}{4} \right)$$

$$= \frac{\pi}{3} + \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$$